PLANE GEOMETRY

Philip D. Gendreau, N. Y.

What makes this picture so beautiful? Cover the reflection. Is it still beautiful? Can you find a line along which the picture might be folded so the two parts would match? This is the axis of symmetry.

PLANE GEOMETRY

BY

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PREFACE

The preparation of the material used in this book was begun more than ten years ago. It grew out of the author's need for a set of exercises comprehensive enough to meet every demand in his classroom and graded according to the difficulties which the pupils experienced in solving them. The exercises were grouped under the proposition with which they were to be used, and notes were added to indicate the time when particular methods should be stressed. For several years the author had no idea of writing a text-book and worked merely to devise and perfect original exercises for use in his classes. Later he collected and graded those he found most effective in pupil-training.

These exercises are the nucleus around which this text-book has been written. Four years have been spent in testing, experimenting with, and in writing the other parts of the book. The reports of all of the prominent mathematical committees have been studied, and the personal advice of some of the foremost mathematical authorities in the United States was obtained. All of the best modern ideas on the teaching of geometry have been considered and applied, and these have been combined with the new ideas and methods which the author's own experience and research have disclosed.

Some of the special features of the book are:

1. *It meets accepted standards.* The list of propositions follows the recommendations of the National Committee on Mathematical Requirements and of the College Entrance Examination Board. The body of the text contains the propositions recommended by these two bodies with only

such others as are needed to make a connected and logical sequence.

(*a*) All propositions on the fundamental list of the National Committee are printed in blackface type so that they stand out easily from the rest of the text.

(*b*) All propositions on the required list of the College Entrance Examination Board are starred.

(*c*) All others are printed in italic.

2. *A minimum course is offered.* The propositions in the body of the text constitute a minimum course. The supplement contains additional material which may be used if time permits, or if called for by variations in courses of study.

3. *There is provision for individual differences.* The emphasis throughout the book is laid upon the exercises rather than upon the formal proposition. The exercises are divided into four groups:

(*a*) Class Exercises. These are exercises for the whole class, and are part of the regular assignment. These exercises in addition to the required propositions provide a minimum course.

(*b*) Optional Exercises. These exercises are intended to be used with classes of high average ability, and for average classes capable of doing more work.

(*c*) Honor Work. In this section are placed exercises for the better pupils of a class.

(*d*) Applied Problems. From these the teacher should select exercises to motivate the course and to add interest generally to the work. These problems are grouped with the proposition to which they apply so that the teacher can tell when to use them.

The applied problems have been selected with the interests of boys and girls in mind. Among them are problems in finding distances to inaccessible points, applications to carpentry, bridge-building, gardening, dress-making, china-painting, real estate, astronomy, radio, aviation, and base-

ball. The problems have been carefully correlated with other high school subjects, and especially with physics.

4. *Complete proofs are given for all important propositions.* The large number of original exercises prevents memorizing, and the fact that reasons in the proof are referred to only by paragraph numbers incites the pupil to think out his reasons for himself if possible, rather than to turn back in the book to search for them. This plan is particularly valuable in that it offers the student who has been absent a means of making up his work, and serves the other students as a model upon which to base their work. The proposition with the full proof is almost a necessity in review work.

5. *Reasons are required for all constructions.* It is well known that faulty construction is the greatest source of error in geometry. Requiring reasons for every construction will eliminate such statements of pupils as: "Construct AB bisecting $\angle C$ and perpendicular to DE."

6. *There is a thorough treatment of motion and functional relationships.* The ideas of motion, variation, and functional relationship have been carefully developed in this text. These concepts are particularly emphasized in such subjects as that of the measurement of angles by arcs and in the area propositions, but they are not limited to these. The idea has been introduced wherever the variation of one quantity can be made to depend upon that of another. Few groups of exercises can be found in the book that do not contain some problems in functional relationship. Loci are first defined in terms of motion, and this concept is carried through all of the exercises.

7. *There is an unusually thorough treatment of numerical trigonometry.* Numerical trigonometry is given more than the grudging page or two in the supplement, which is too often accorded it. Since few pupils ever take a course in trigonometry, it is almost necessary to teach the fundamental principles of surveying and indirect measurement in a

geometry course. This book provides generously for this necessity.

8. *Methods of thinking are emphasized.* Methods of attack and the mechanics of thinking out a proof are emphasized throughout the book. Each new method is called to the attention of the pupil as soon as he has used it, and suitable exercises are provided for further practice in using it. Problem analysis is carefully explained.

9. *Investigation problems are a new and original feature.* These problems, coming before the important propositions, lead the pupil to discover the proposition and to work out the proof for himself. The teacher may use the investigation problem as a basis for developing the proof of a proposition in class, before the study of the proposition has been taken up.

10. *Self-measuring tests.* At important points in the book, the pupil is given "Self-measuring tests" which help him to estimate his grasp of the material he has studied. At the end of each book there is a summary of the important ideas learned in that book. This material may serve for review purposes in addition to the Review Problems.

11. *There is a systematic grouping of logically related topics.* The propositions have been arranged so that the pupil gets the idea of a continuous group of propositions built up into a logical whole.

12. *Only terms and symbols recommended by the National Committee are used.*

13. *Essential parts of the hypotheses are emphasized.* In order to help prevent an often recurring situation in which students fail to remember the essential part of the hypothesis, the facts that are readily observable from the figures are set off in brackets. The author recommends that the student's attention be called especially to that part of the statement that is not within the brackets.

14. *New-type tests and historical material* have been included, the latter in an original manner intended to stimu-

late the interest of the pupil in the historical phase of the subject.

The author acknowledges his indebtedness to many persons who have contributed to this volume by their advice and assistance, and in particular to Walter F. Downey, Head Master of the English High School, Boston, for his careful reading of the manuscript and for helpful advice and suggestions throughout the work; to John W. McCormack, Head of the Department of Physical Science in Jamaica High School, New York City, for many of the exercises correlating mathematics with science; to Emily J. Haddon, for exercises of interest to girls and for reading the proof; and to William Betz, supervisor of junior and senior high school mathematics in Rochester, N. Y., and Walter Roberts, of West Philadelphia High School, Philadelphia, for reading the manuscript before publication and for many helpful suggestions.

Special acknowledgment is due Dr. Raleigh Schorling of the University of Michigan, who read the manuscript and the page proof and contributed the new-type tests and material on the history of geometry. By his detailed work he insured rigorous mathematical treatment. He also suggested numerous learning devices applying principles drawn from educational psychology and refined by years of observation of school practice.

J. P. McC.

PREFACE TO THE SECOND REVISED EDITION

At no time has the subject-matter of high-school mathematics undergone such close scrutiny and have such rapid advances been made in fitting it to the life needs of the pupil as during the last few years. In geometry, the tendency has been to eliminate those parts that have no other worth than that of information and to substitute in their place material that better meets the major aims of mathematical instruction, training in reasoning, in spatial imagination, and in those types of thinking likely to be most useful in life situations. In accord with these aims, the following changes were made in the first revision.

1. The number of exercises has been increased on important topics that the teacher may desire to stress, such as congruent triangles, loci, and similar triangles, which, with the great amount of original work the earlier edition grouped around a small number of book theorems, enable the teacher to center emphasis on original thinking rather than on memory.

2. A large amount of construction has been placed at the beginning of the text so that the pupil may learn the fundamental concepts by doing rather than by memorizing definitions.

3. Intuitive solid geometry has been added at those points at which it is natural to carry the concepts of geometry over into three dimensions. This brings geometry closer to life since we live in a three-dimensional world, and it gives the pupil an opportunity to exercise his spatial imagination to an extent that is impossible in plane geometry.

4. In the belief that much of the reasoning of life is of the indirect type, this topic has been given added emphasis.

5. The number of exercises in the new type tests has been very greatly increased, and these have been classified and distributed so that the busy teacher can use them whenever needed without the necessity of working them in advance to learn if the theorems already studied cover all of the questions.

6. The concept of symmetry has been introduced in pictures showing its applicability to nature and living forms.

7. Geometric reasoning has been directly applied to life situations. This is in keeping with the modern belief of psychologists that the transfer of training is increased if there is a conscious attempt to connect the work of the classroom with the life of the individual.

In the present edition the following changes have been made.

1. The introduction has been livened by new exercises on the line, etc., by applications that make the axioms more real to the pupil, by more and better directed construction exercises, and by life situations to bring out the idea of postulate systems.

2. The measurement of angles by arcs has been improved so as to make it better bring out the idea of the continuous change of a function.

3. A summary of important loci has been added.

4. The topic of inequalities has been improved and added to because it contributes most to the forms of indirect reasoning so common in life.

5. Several pictures with comments showing the value of geometry have been inserted.

6. The application of mathematical reasoning to life situations has been particularly stressed by the further addition of many exercises involving axioms, postulates, ratio and proportion, and the indirect proof.

The author wishes to acknowledge his indebtedness to the many friends who have helped in this work by valuable

suggestions, and in particular to A. I. Barbanell, Thomas Jefferson High School, Brooklyn, N. Y.; Samuel Welkowitz, Franklin K. Lane High School, Brooklyn, N. Y.; Dorothy M. Keeler, Port Chester High School, Port Chester, N. Y.; Murray J. Leventhal, James Madison High School, Brooklyn, N. Y.; Ethel M. Hight, Skowhegan High School, Skowhegan, Me.; Philip R. Dean and Josephine Brand, Evander Childs High School, New York; Henry H. Shanholt, Abraham Lincoln High School, Coney Island, N. Y.; Ida M. Elliott, Riverside High School, Buffalo, N. Y.; Wayne L. French, Shaker High School, Cleveland, O.; Aaron Freilich, Bushwick High School, Brooklyn, N. Y.; Ralph P. Bliss, Alexander Hamilton High School, Brooklyn, N. Y.; H. A. Swineford, Withrow High School, Cincinnati, O.; Norman Collins, Public High School, Scotia, N. Y.; Clara Eaton, Newtown High School, Elmhurst, N. Y.; Alberta S. Wanenmacher, Hutchinson Central High School, Buffalo, N. Y.; Herman L. Lutz, Ellen Nomoff, and Virginia Frey, Theodore Roosevelt High School, New York; Benjamin Braverman, High School of Commerce, New York City; Mildred Frazier, Santa Ana Senior High School, Santa Ana, Cal., G. Phillips, Thornton Township High School, Harvey, Ill.

J. P. McC.

CONTENTS

PLANE GEOMETRY

SYMBOLS AND ABBREVIATIONS

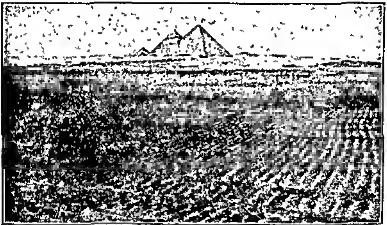
The following symbols and abbreviations are used in this book.

$\angle$	angle
$\measuredangle$	angles
$\triangle$	triangle
$\triangle$s	triangles
$\perp$	perpendicular, or is perpendicular to
$\parallel$	parallel, or is parallel to
$\cong$	congruent, or is congruent to
$\sim$	similar, or is similar to
$>$	is greater than
$<$	is less than
▱	parallelogram
$\odot$	circle
Ⓢ	circles
$\frown$	arc
A'	A prime

...	and so forth
alt.	alternate
ax.	axiom
const.	construction
corr.	corresponding
ext.	exterior
hyp.	hypothesis
iden.	identity
int.	interior
isos.	isosceles
post.	postulate
rt.	right
st.	straight
subst.	substitution
supp.	supplement
vert.	vertical

INTRODUCTION

1. The geometry of the ancients. The name *geometry* is derived from two Greek words meaning "earth measure." The Greeks gave this name to the subject that you are about



Elmendorf photo, © Ewing Galloway

MORE THAN 3,000 YEARS AGO THE EGYPTIANS WORKED OUT A CRUDE SYSTEM OF GEOMETRY.

to study because the Egyptians, from whom they learned the subject, used it in surveying. It is a very old science and grew out of the needs of the people.

Every year the Nile River overflowed its banks, spread out over the flat plains of Egypt, and washed away many landmarks that indicated the limits of the fields bordering on the river. In order that the boundaries might again

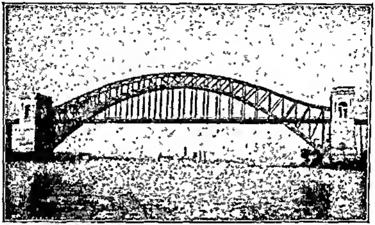
be set accurately, the Egyptians more than three thousand years ago worked out a crude system of geometry. This geometry consisted merely of a few rules for surveying, which, through long experience, they had developed. They did not ask themselves why these rules should be true, but were satisfied to use them when it was found they gave fairly satisfactory results.

It is probable also that other early nations, such as the Babylonians, worked out a few simple rules for surveying and for constructing buildings.

Long afterwards, but more than five hundred years before the time of Christ, the Greeks became interested in Egyptian geometry. The Greeks were a thinking people, and sought reasons for these rules. Gradually they built up the geometry which we study to-day, and, in so doing, they found that many of the rules used by the Egyptians were not strictly true. For example, the Egyptians found the area of an isosceles triangle by multiplying one of the legs by half the base. This, you will learn later, often gives a result that differs by a large amount from the correct value. It never gives the exact area.

Uses of geometry to-day. The Greeks cared more for the theory, and in Euclid's great book on geometry, written about 300 B.C., there are no practical applications. Geometry, however, has many applications. The surveyor measuring land or laying out streets, and the civil engineer constructing a railroad, a bridge, or a building depend upon it. Also the physicist, in his work in mechanics, sound, light, or electricity, the astronomer, studying the stars and determining the correct time for our clocks, the navigator on the sea and in the air finding his position, the draughtsman, and the mechanical engineer all need geometry. Its principles are continually used, too, by estimators for excavations, for brick work or stone work, for monuments, and for plumbing and plastering, as well as by many other people.

You will find much of this geometry very simple, so simple, in fact, that you will ask yourself why we waste time learning



Photograph by H. A. Webb

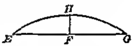
GEOMETRY PLAYS AN IMPORTANT PART IN BRIDGE CONSTRUCTION.
The strength and rigidity of this bridge depend on the triangle and the circular arc. The picture shows Hell Gate Bridge, New York.

to prove things that are already so evident. But we must start somewhere, so we start with facts so simple that everyone will admit their truth. As we proceed, each part becomes just as simple, but, if we should attempt at the beginning to solve the problems in the middle of the book, we should find the task impossible, without having had the first part.

Here, for example, are illustrations of what you will meet later in the course. A surveyor finds the distance across a river from A to B by measuring the length from B to C, and the number of degrees in the angles at B and C.

Again, the railroad engineer must know the radius of every curve so that he can raise the outside rail enough to keep the train from jumping off the track. Usually he cannot

measure the radius directly, especially if the curve be built around a ledge or a bend in a river. When this is the case, he measures from E to F, from F to G, and from F to H. From these measurements, he finds the radius he needs.



Such problems as these and many others that someone must solve in his daily work, you will learn in this book. Before you can understand them, however, you must understand the principles involved in the easier exercises that come before them.

It may happen that you will never need to solve problems like those given above. Of what value, then, is geometry to you? The real value of geometry is not so much in the facts about space and design that it teaches you, interesting as these may be, but in the training it gives you in logical thinking. It will teach you the way to discover new facts for yourself and how to convince others of facts that you know. It will train you to back up your assertions with reasons and to weigh the statements made by others. It will help you to express your thoughts in clear language that will mean what you want it to mean.

But in order that it may do these things for you, you must attack it in the right way. Form the habit of thinking out things for yourself. When you read a statement to be proved, stop and see how much of it you can do without reading the proof from the book. Memorizing the proof in the book will be of very little value to you. You wish to develop the power to think better, and this can be done only by thinking. If you train yourself to think out things without help, you will find geometry both interesting and valuable, but if you memorize the proofs, it will prove to be a dull subject, and you will derive little benefit from it. Be honest with yourself. You will be the gainer.

We shall begin by recalling a few facts about points and

lines which you already know. So much of our work deals with these that it will be necessary to form very clear ideas about such fundamental things before beginning the proofs. Consequently, you should do these exercises very carefully.

Exercises

1. How thick is a line? If you wished to represent a line, would you use a stick an inch thick, a pencil, a wire, or a piece of spider's web? Why? Can you think of a line that has no width at all? With your pencil draw a line on paper. Is this really a geometric line? Why not? Would the boundary between the black of this line and the white of the paper be more nearly a line?

A geometric line has no width or thickness.

2. How long is a line? Can you think of a line a mile long? Ten miles long? A thousand miles long? Can you imagine a line without any ends, going on and on, beyond the sun, beyond the stars?

A geometric line is unlimited in length. Any part of it is called a line segment.

3. What is a straight line? Look along the edge of your ruler. Can you make the whole edge look like a single point? Could you do this with a curved line? How then can you tell a straight line from a line slightly curved?

4. What is a point? Draw a very light line on your paper. Remember that your line only represents an ideal line which has no width. Now draw another line crossing the first. How large is the point where they cross? If your lines had no width, how large would this point be? There is a period at the end of this sentence. Is it a point? Why not? If it were smaller, would it be a point? Can you think of it as a small circle whose center is a point?

2. A point is represented by a dot, and is read by a capital letter placed near the dot, as point A. A geometric point has no size. $\cdot A$

5. How many points are needed to determine a straight line, that is, how many points of a straight line must I name so that you can tell which line I am talking about?

(a) Mark a point on your paper. Now draw a line through it. Can you draw another line through the same point? Would the one point be enough to tell which of your lines was meant?

(b) Does one point determine a straight line?

(c) Mark two points on your paper and draw a straight line through both. Can you draw another straight line through both points? How many different straight lines can be drawn through the same two points?

(d) Do two points determine a straight line?

6. Why has a gun two sights? Could you hit an object if only one of the sights was in the line from your eye to the object?

7. Draw two different straight lines through a point. Will these lines meet in another point if extended?

3. A straight line is represented by a stretched thread or by the edge of a ruler. It is fixed in position by any two of its points. Therefore:

4. *Only one straight line can be drawn through two points.*

5. *Two straight lines can meet in only one point.*

A straight line is usually denoted by two capital letters, one at each end of the line segment drawn, but it may be designated by reading any two of its points. A line segment of definite length is often designated by reading a small letter placed near it. Thus the line AB; the line CD; the line a.

Where no confusion can result, a straight line or a line segment is called simply a *line*. A geometric line has neither width nor thickness. It has only one dimension, length.

EXERCISES

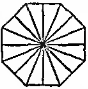
1. What does a radio salesman mean when he says, "This set will receive all stations within a radius of 1,000 miles?" If you had a map of North America with a scale of miles, could you draw

a line which would include all stations within 1,000 miles of your home and no others?

2. What is the line in Ex. 1 called? How far is it from your home to any point on this line? How does the distance from your home to one point on it compare with the distance from your home to any other point on it?

3. If a city is less than 1,000 miles from your home, will it be inside, on, or outside this line? If more than 1,000 miles?

4. Why are the wheels on a wagon made circular instead of square or shaped as in the figure?



5. Why would the wagon jar if the wheels were not round?

6. Draw a circle on paper. Now draw a straight line which passes through the center of this circle. In how many points does it touch the circle?

7. Can you draw a straight line which will touch the circle in only one point and which will not touch it again no matter how long you make it?

8. Can you draw a straight line which crosses your circle in 3 points? What is the greatest number of points in which you can make a straight line cross a circle?

9. Draw two circles which cross each other. In how many points do they cross?

10. Draw two circles which touch in only one point. Now draw a straight line which touches one of them at this point but does not enter the circle. In how many points does it touch the other circle?

11. Draw a circle entirely inside another circle and not touching it. Can you draw a straight line which will touch each of them in only one point no matter how far you extend it?

12. Can you draw two circles which cross in three points? In four points? What is the greatest number of points in which two circles can cross each other?

6. A plane is a smooth, flat surface. The surface of a pane of window glass, of still water, and of the walls of the

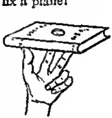
room are good examples of planes. If the straight edge of a ruler is placed against a plane in any position so that its ends touch the surface, the whole edge will touch the surface. Plasterers test their work in this way.

A plane is often defined as a surface such that a straight line joining any two of its points lies wholly in the surface. A plane has length and width, but no thickness.

Space Geometry (*Optional*)

1. You have learned that not more than one straight line can be passed through two points. Do you think that more than one plane could be passed through two points? Through three points on a straight line? Through three points not on a straight line?

2. Hold one finger up vertically and support your book horizontally on it. Can you tip the book without removing it from your finger tip? Does one point fix a plane?



3. Hold up two fingers apart and try the same experiment. Can you still tip the book without removing it from either finger? Do two points fix a plane?

4. Hold up three fingers with the tips not in a straight line. Can you still tip the book without removing it from any finger? Do three points fix a plane?

5. What is the smallest number of points needed to hold a plane in position? Will these points hold the plane fixed if they are in a straight line?

6. Is a plane fixed in position by a point and a straight line? By two straight lines which meet in a point?

7. Can you think of two straight lines in space through which it would be impossible to pass a plane? Illustrate with two rulers or two pencils or two edges of the class room.

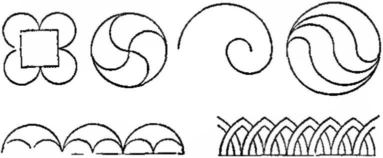
8. Fold a sheet of paper. Is the edge of the fold a straight

line? Try to make a curved crease. Do both parts of the paper remain plane? Do two planes always cut each other in a straight line? Is it possible for two planes to have a curved line in common? Give a reason for your answer.

7. A **circle** is a closed plane curve, all points of which are the same distance from a fixed point in the plane called the *center*. Any part of the circle is an **arc**. The **radius** is a line from the center to any point on the curve. Therefore, all radii of a circle are equal. A **chord** is a line joining two points of the circle. A chord passing through the center is a **diameter**.

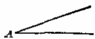
DRAWING EXERCISES

Copy the following designs. Notice that the spiral shown here is made up of half circles.



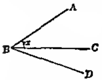
8. An **angle** ($\angle$) is formed by two straight lines which meet at a point. The lines are the **sides** of the angle, and the point is its **vertex**.

An angle is read by a capital letter placed near its vertex, as $\angle A$. However, when two or more angles have the same vertex, this method of reading is open to misunderstanding. It is then desirable to place a small letter within the angle, as $\angle x$.



Or an angle may be read by three letters, one at the vertex and

one on each side, the letter at the vertex being read between the other two, as $\angle ABD$ or $\angle DBA$. The letters are read in the order in which one would come to them, were he to draw the angle with one continuous stroke of the pencil.

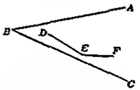


We may think of $\angle ABC$ as having been formed by turning a line around point B as a pivot from the initial position BC to the final position BA. The size of the angle depends on the amount of this turning and not on the length of the sides.

ILLUSTRATION. Lay your book closed on the desk. Now open the front cover without moving the book. Notice that the cover turns around the back edge as the book opens. The size of the angle which the lower edge of the cover makes with the lower edge of the first page depends on the amount the cover has turned.

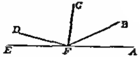
EXERCISES

1. Draw two lines AB and CD crossing at point E. Read the angles formed by these lines.



2. Which of the two angles ABC and DEF is the larger? If you lengthen the lines BA and BC, does $\angle ABC$ become larger?

3. Which is larger, $\angle AFB$ or $\angle AFC$?



4. If $\angle CFD$ is added to $\angle AFC$, what angle is the result?

5. If $\angle AFB$ is subtracted from $\angle AFD$, what angle is the result?

6. Is there any angle in the figure larger than $\angle AFD$?

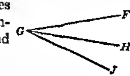
7. If $\angle DFE$ is added to $\angle AFD$, what angle is the result?

8. Draw a straight line XY. Now draw another line RS meeting XY at S so that two equal angles are formed. What is the name of this kind of angle?

9. Draw an angle smaller than one of the angles in Ex. 8. What is this angle called?

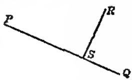
10. Draw an angle larger than one of the angles in Ex. 8, but smaller than the sum of the two equal angles. Name this angle.

9. Adjacent angles are two angles which have the same vertex and a common side between them; as $\angle FGH$ and $\angle HGJ$.



10. A straight angle is an angle whose sides lie in a straight line in opposite directions from the vertex; as $\angle KLM$.

11. If one straight line meets another straight line so as to make two adjacent angles equal, each of these angles is a right angle. $\angle PSR$ and $\angle RSQ$ are right angles.



Show that a right angle is one half of a straight angle.

12. Two lines meeting at right angles are perpendicular ($\perp$) to each other; as PQ and RS.

A definition is always reversible. Consequently § 12 may be quoted in the form: Two lines perpendicular to each other meet at right angles, or perpendicular lines form right angles.

Show that perpendicular is not the same as vertical.

The foot of the perpendicular is the point where the lines meet; for example, point S.

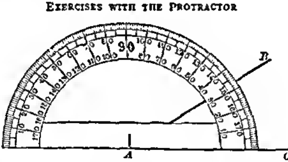
13. An acute angle is an angle smaller than a right angle; as $\angle A$. An obtuse angle is an angle larger than a right angle, but smaller than a straight angle; as $\angle B$ in the figure below.

14. The degree. The ordinary unit for measuring angles is the *degree* (°), which is one-ninetieth of a right angle. The degree is divided into sixty equal parts called *minutes* ('), and the minute is again divided into sixty equal parts called *seconds* (").

$60'' = 1'$
$60' = 1°$
$90° = 1$ rt. angle
$180° = 1$ st. angle

A degree is also divided into tenths, hundredths, and thousandths; that is, a part of a degree is expressed as a decimal fraction as 68.374°.

Exercises with the Protractor



Protractor

Note. A protractor is an instrument for measuring angles. It is made in the form of a half circle whose arc is divided into 180 equal parts called *degrees*. An angle is measured by placing the protractor so that the vertex of the angle is at the center of the circle, and one side of the angle runs along the straight edge, as shown in the diagram. The number of degrees in the angle is read at the point where the other side crosses the scale.

1. Draw an acute angle *BAC*. Now using a protractor as shown in the figure on the previous page, measure the number of degrees in your angle.

2. Draw an angle as nearly a right angle as you are able without using the protractor or compasses. Now measure it with the protractor, and note how many degrees it differs from a true right angle.

3. Similarly, draw and afterwards measure angles of 45°, 135°, 30°, 60°, 120°, $22\frac{1}{2}°$, $67\frac{1}{2}°$, 20°, and 40°. In each case compare your estimate with the measured result.

4. Draw an angle whose sides are each two inches long, and measure it. Now prolong the sides, making one of them three inches and the other four inches long. Measure the angle again. Has its size changed?

5. Draw two angles as nearly equal as you can estimate. Measure them, and determine the number of degrees by which they differ.

6. Draw an angle *ABC*. Now draw a line *BD* which you think will divide $\angle ABC$ into two equal parts. Measure the two parts, and determine their difference.

7. Draw an angle of 45°, and extend one of the sides through the vertex. Measure both angles. Find their sum.

6. Make the same drawing and measurements, starting with angles of 135°, 30°, 60°, 120°, 150°, $22\frac{1}{2}°$, $67\frac{1}{2}°$, 20°, and 40°. Could you have computed the number of degrees in each of the angles formed by extending the line, without measuring?

9. Draw two lines which cross each other. Measure two opposite angles. How do they compare in size? Also measure another angle of the same figure. What is the sum of this angle and either of the other two?

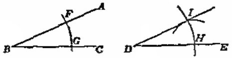
10. Draw two angles that have the same vertex but are not adjacent. Draw two angles that have a common side but are not adjacent. Explain.

How to copy an angle. Architects make an angle equal to another angle by using only the compasses and a straight-edge ruler. Can you do it without reading the directions below?

To make an angle equal to angle B:

1. Place the point of your compasses on B and with any convenient opening make an arc cutting AB at F and BC at G.

2. Keeping the same opening of the compasses, put the point on D and draw an arc cutting DE at H.



3. With your compasses measure GF. Then put the point on H and make an arc cutting the other arc at I.

4. With your ruler draw DI.

$\angle D$ will equal $\angle B$.

Exercises with Compasses and Ruler

1. With your protractor draw an angle of 70°. Then with ruler and compasses construct an angle equal to it. Check with your protractor. (We construct a figure when we draw it accurately using straight-edge ruler and compasses only.)

2. Draw the following angles and construct an angle equal to each: (a) 15°, (b) 110°, (c) 90°, (d) 45°.

3. Draw an angle of 35°. Now construct an angle twice as large.

4. Draw any two angles. Then construct an angle equal to their sum.

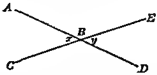
5. Construct an angle that is equal to the difference of two given angles.

6. Draw an angle of 20°. Construct an angle, (a) three times as large; (b) four times as large; (c) six times as large.

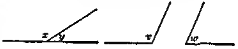
7. Draw three unequal angles and construct an angle equal to their sum. Check with your protractor.

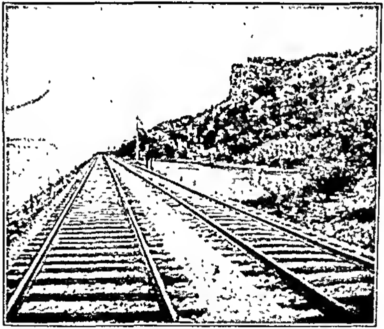
8. With your protractor draw two angles, one of 30° and the other of 45°. Now by using combinations of these two angles construct with compasses and ruler angles of (a) 75°, (b) 60°, (c) 105°, (d) 15°, (e) 90°.

15. When two straight lines cross, the opposite angles are called vertical angles; as $\angle x$ and $\angle y$.



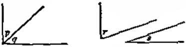
16. Supplementary angles are two angles whose sum is a straight angle; as $\angle x$ and $\angle y$, or $\angle v$ and $\angle w$. Each angle is the supplement of the other.





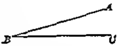
RAILROAD TRACKS ARE A FAMILIAR EXAMPLE OF PARALLEL LINES

17. Complementary angles are two angles whose sum is a right angle; as $\angle p$ and $\angle q$, or $\angle r$ and $\angle s$.



CLASS EXERCISES[1]

1. If the line AB turns around point B in the direction opposite to that in which the hands of a clock move until it lies in a straight line with BC, name in order the four kinds of angles it forms with BC.



2. What angle equals the sum of a right angle and an acute angle? What angle equals their difference?

3. If an acute angle is subtracted from a straight angle, what kind of angle is the result?

4. If an acute angle is subtracted from a right angle, what kind of angle is the result?

5. Compare an acute angle with its supplement.

6. If an acute angle grows larger, how does its supplement change? How large must the angle become to equal its supplement? To exceed its supplement?

7. Name the angle which is two-fifths of a straight angle. Three-fifths of a straight angle.

8. Is half an obtuse angle necessarily an acute angle? Is twice an acute angle necessarily an obtuse angle?

9. If three times an acute angle is an obtuse angle, what number of degrees must the acute angle exceed?

[1] Enough work has been provided under the heading "Class Exercises" for the average class. However, extra exercises of a somewhat more difficult nature have been included, captioned "Optional Exercises." These are intended for classes which have sufficient time for a greater number of exercises than are given under the first heading. The third division, "Honor Work," contains exercises of a degree of difficulty to test the mettle of the better pupils.

10. What is the angle formed by the hands of a clock at 3 P.M.? At 6 P.M.? At 4 P.M.? At 2 P.M.?

11. Through how many degrees does the minute hand of a clock turn in one minute of time?

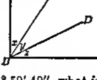
12. How many degrees are there in the angle formed by the hands of a clock at 12:30 P.M.? At 2:30 P.M.?

13. Through how many degrees does the minute hand pass in 20 minutes? In 13 minutes?

14. If the earth turns completely around in 24 hours, through how many degrees does it turn in 1 hour? In 2 hrs. 15 min.?

15. What angle is $\angle ABE + \angle EBD$? $\angle EBC - \angle DBC$?

16. If $\angle ABC = 90°$, what is the complement of $\angle ABD$? Of $\angle x$?



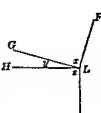
17. If $\angle ABC = 90°$ and $\angle ABD = 62° 10'$, find the number of degrees in $\angle z$.

18. If $\angle ABC = 90°$, $\angle x = 35° 20'$ and $\angle y = 28° 15'$, find $\angle z$.

19. If $\angle ABD = 58° 40' 32''$ and $\angle y = 20° 15' 25''$, find $\angle x$.

20. If $\angle ABC$ is a right angle and $\angle y = 21° 50' 40''$, what is the sum of $\angle x$ and $\angle z$?

Optional Exercises



21. $\angle x$ and $\angle z$ are right angles and $\angle y = 20°$. How many degrees are there in $\angle FLK$? In $\angle FLH$?

22. If $\angle x$ and $\angle z$ remain right angles, but $\angle y$ increases from $20°$ to $100°$, describe the change which takes place in $\angle FLK$.

23. How many degrees are there in $\angle y$ when it becomes equal to $\angle FLK$?

24. Find the supplement of $50°$. Of $67° 18'$. Of $112° 43' 32''$.

25. If the complement of an angle of $x°$ contains $2x°$, find x.

26. Find that angle which is $\frac{3}{4}$ of its supplement.

27. The greater of two supplementary angles exceeds the smaller by $22°$. Find both angles.

28. By how much does the supplement of an angle exceed the complement of that angle?

29. If 34° were added to an angle, it would then equal the supplement of the original angle. Find the number of degrees in the original angle.

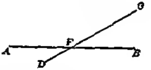
30. An angle exceeds its supplement by 38°. How many degrees must be subtracted from it in order that it equal its supplement?

31. Draw a pair of adjacent angles which are supplementary. A pair of adjacent angles which are not supplementary. A pair of supplementary angles which are not adjacent.

COMMON TERMS

18. Lines intersect if they have one or more common points.

19. To bisect is to cut in two equal parts, or in halves.



AB intersects *CD* at *F*.
CD bisects *AB* at *F*, if $AF = FB$.

20. Figures are said to coincide, if one of them exactly fits the other.

21. (*a*) Figures which can be made to coincide are congruent ($\cong$).

(*b*) Any two figures are congruent if all the parts of one equal respectively the corresponding parts of the other. For, in that case, they could be made to coincide.

22. *The corresponding sides and angles of congruent figures are equal.*

How to bisect a line segment.

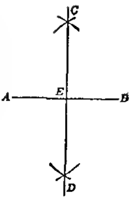
To bisect AB:

1. With A as center and with a radius more than half of AB, construct arcs on both sides of AB.

2. With B as center and with the same radius, construct arcs cutting the first arcs at C and D.

3. With your ruler draw CD cutting AB at E.

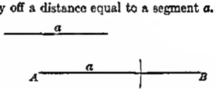
4. E is the middle point of AB.



NOTE. CD is also perpendicular to AB. It is called the perpendicular bisector of AB.

CONSTRUCTION EXERCISES

1. On a given line AB, lay off a distance equal to a segment a. With an opening between the points of the compasses equal to the length of a, and with A as center, construct an arc cutting AB.



2. Construct a segment three times as long as a given segment.

3. Construct a line segment equal to the sum of two unequal segments a and b.

4. Draw a vertical line segment and bisect it. Test with your compasses.

5. Draw several segments of different lengths and in different directions. Bisect them and test with your compasses.

6. Draw a line segment. Then construct a segment $1\frac{1}{2}$ times as long.

7. Divide a line segment into 4 equal parts.

8. By the method learned above can you divide a line segment into 3 equal parts? 5 equal parts? 8 equal parts? 10 equal parts? Construct those that are possible.

9. Construct a line segment equal to the sum of three given line segments.

10. If a and b are two unequal segments, a being longer than b, construct $a+b$; $a-b$; $2a+3b$; $2a-b$.

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Draw an angle ABC. With B as center, draw an arc cutting AB at D and BC at E. Using D and E as centers and with the same radius, draw two arcs crossing at F. With your ruler draw a straight line from B to F. BF bisects $\angle ABC$.

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1. Bisect an angle of 45°; 60°; 90°; and 120°. Test each angle with your compasses or measure with your protractor. Does the line always cut the angle into two equal parts?

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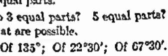
8. Construct a square having a given line segment as side.



9. This design was used on a church in Zurich. Begin with a line segment KL and construct 4 squares. Then use their corners as centers for arcs.

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Now geometry is a study in reasoning. Certain facts are given us. From these we draw a conclusion. We prove that a statement is true if the data given are true. To make sure that we shall not slip in our reasoning, we follow a definite form, giving a reason for every step in the process.

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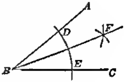

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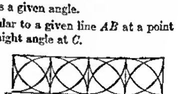

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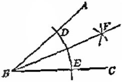

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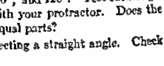

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Exercises

In the following statements, tell which part is the hypothesis and which the conclusion.

1. If there were no air, there would be no twilight.
2. Water freezes if it is cooled below 32° Fahrenheit.
3. A book which is well bound will last longer.
4. If two triangles are congruent, their corresponding parts are equal.
5. An airplane will travel more slowly if the wind is against it.
6. A line which passes through the center of a circle cuts the circle in two points.
7. The base angles of an isosceles triangle are equal.

Axioms and Postulates

28. It is evident that, if we are to give a reason for every step in a proof, we must agree on some facts which we can use as reasons in our first proofs. We shall therefore begin by making a list of simple statements which all will accept. Such simple statements are called *axioms* and *postulates*.

29. An axiom is a statement accepted as a truth without proof. A postulate is a construction admitted as possible without proof.

30. Axioms and geometric assumptions.

On the right pan of a balance are 10 lbs. of weights, on the left a can of sugar. How many pounds of sugar are there?

If I replace two 5-lb. weights by a 10-lb weight, will the scale still balance? Can I always substitute an equal weight without disturbing the balance?

Axiom 1. In any process, a quantity may be substituted for an equal one (called "substitution").

If I replace the can of sugar by a can of coffee and the scale still

balances, how will the weight of the coffee compare with that of the sugar?

Axiom 2. Things equal to the same thing or to equal things are equal to each other.

If I now pour 2 lbs. of coffee into the can on the left pan, what must I do to the right pan to keep the balance? If I have equal amounts on the two pans, must I always add equal amounts to keep the balance?

Axiom 3. If equals are added to equals, the results are equal.

If I take 5 lbs. of coffee from the can, what must I do to the right pan to keep the balance? If I have equal amounts on the two pans, and take away equal amounts from both, will the result still balance?

Axiom 4. If equals are subtracted from equals, the results are equal.

If I put three times as much coffee on the left pan, how many times as much weight must I have on the right pan to keep the balance? If I multiply equal amounts by equal numbers, will I always have equal results?

Axiom 5. If equals are multiplied by equals, the results are equal. *Corollary.* Doubles of equals are equal.

If I remove half the weight from the left pan, what must I do to the right pan to keep the balance?

Axiom 6. If equals are divided by equals, the results are equal. *Corollary.* Halves of equals are equal.

A bag of sugar is too large for the scale pan, so I divide it into two parts and weigh each separately. One part weighs 13 lbs. and the other weighs 8 lbs. What is the weight of the whole bag of sugar? Do you always add the parts to get the weight of the whole?

Axiom 7. The whole equals the sum of all its parts.

Does either of the parts into which we divided the bag of sugar weigh as much as the whole bag of sugar weighs? Is it always true that the whole amount weighs more than one of its parts?

Axiom 8. The whole is greater than any of its parts.

Geometric assumptions.

9. A straight line is the shortest line joining two points.

10. A geometric figure can be moved without change of size or shape.

11. Through a point not more than one line can be drawn parallel to a given line.

12. The perpendicular is the shortest line from a point to a line.

13. Every angle has a bisector.

14. A line segment has a middle point.

31. Postulates.

1. A straight line can be drawn through any two points.
2. A straight line can be extended as far as desired.
3. A circle or arc can be constructed with any center and any radius.

The postulates tell us that in construction we are allowed to use the straightedge ruler and the compasses only. Other instruments such as the protractor, the triangle, or the square we shall find useful in ordinary drawing, or in checking the accuracy of our work, but we must be careful never to use them in geometric construction.

Exercises

1. Give the axiom that best fits each of these statements:

(*a*) If a pound of butter costs 32 cents, 3 lbs. of butter will cost 96 cents.

(*b*) If a pound of butter costs 32 cents and a pound of sugar costs 6 cents, then a pound of butter and a pound of sugar will cost 38 cents.

(*c*) If a pound of butter costs 32 cents, a quarter of a pound of butter will cost 8 cents.

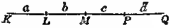
(*d*) If a pound of butter costs 35 cents and a pound of coffee costs 35 cents, the grocer ought to be willing to exchange a pound of coffee for a pound of butter.

(e) I bought a pound of butter for 35 cents. Mrs. Rice wants only a part of a pound of butter. She ought to get it for less than 35 cents.

(f) I bought some groceries, including a loaf of bread, for which the grocer wants 80 cents. I decided then not to take the bread which was worth 9 cents. So the grocer ought to charge me only 71 cents.

2. If $a=b$ and $b=c$, why does $a=c$?

3. If $a=b$, $c=d$, and $b=d$, why does $a=c$?



4. If $a=c$ and $b=d$, why does $a+b=c+d$?

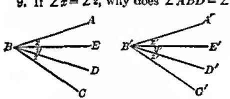
5. If $a=c$ and $b=d$, why does $a-b=c-d$?

6. If $a=c$, why does $a+b=b+c$?

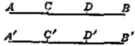
7. If $KM=LP$, why does $a=c$?

8. If $KM=MQ$ and $a=d$, why does $b=c$?

9. If $\angle x=\angle z$, why does $\angle ABD=\angle EBC$?



Exs. 9, 10, 11, 12.



Exs. 13, 14.

10. If $\angle ABD=\angle EBC$, why does $\angle x=\angle z$?

11. If $\angle B$ and $\angle B'$ are each cut into three equal angles, x, y, z and x', y', z' respectively, and $\angle x=\angle x'$ why does $\angle B$ equal $\angle B'$? (B' is read B prime.)

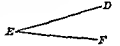
12. If $\angle x=20°$, $\angle y=25°$ and $\angle z=30°$, how many degrees are there in $\angle B$? Why?

13. If AB is cut into three equal parts at C and D, and $A'B'$ is cut into three equal parts at C' and D', and $AC=A'C'$, why does $AB=A'B'$?

14. If AB and $A'B'$ are cut into three equal parts at C, D, C', and D' respectively, and AB equals $A'B'$, why does $CD=C'D'$?

15. If FG is 2 in. long and GH is 4 in. long, how long is FH? Why?

To construct a straight line in each of the following cases, state how many points of that line must be found:



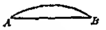
16. The straight line perpendicular to FH and bisecting FH.

17. The straight line perpendicular to FH at point G on FH.

18. The straight line bisecting $\angle DEF$.

19. Why is BC shorter than $BA+AC$?

20. Why is chord AB shorter than arc AB?



Life Situations. (*Optional*)

In business advertising, many assumptions are made. These are seldom stated in words, and are often false.

21. Name at least one assumption that each of these advertisers wishes you to make. Do you think this assumption is true?

(*a*) Menin will relieve you quickly. See how quickly it dissolves.

(*b*) Rubiden is the only tooth paste containing murien.

(*c*) Do as millions of other people do. Use vitatone.

(*d*) The elephant cigarette is his majesty's blend.

(*e*) Armo tea is blended.

22. What assumption is made in these two illustrations? Are the two cases equally likely?

(*a*) You put off studying until sunrise tomorrow. You are sure the sun will rise because it has risen every morning for the past 1000 yrs.

(*b*) Italians live on the slopes of Mt. Vesuvius without fear. Their ancestors have lived there safely for the past 1000 yrs.

23. Examine advertisements in magazines or cars. Can you find the assumptions that the advertiser wishes you to make? Are these assumptions true?

Simple Theorems

32. *All straight angles are equal.* (See § 4.)

33. *All right angles are equal.* (See § 32 and Ax. 6.)

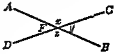
34. *Supplements of the same angle or of equal angles are equal.* (See § 32 and Ax. 4.)

Also, complements of an angle or of equal angles are equal.

35. *If two adjacent angles have their exterior sides in a straight line, they are supplementary.* (§§ 9, 16.)

36. *The sum of the angles about a point is two straight angles.* (Draw a straight line through the point.)

37. *Vertical angles are equal.* $\angle x$ and $\angle z$ are each supp. to $\angle y$ (§ 35). Therefore $\angle x = \angle z$ (§ 34).



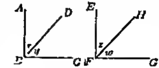
Class Exercises

1. If $AB \perp BC$, $EF \perp FG$, and $\angle x = \angle z$, show that $\angle y = \angle w$.

2. If $KQ \perp MQ$, $LQ \perp NQ$ and $\angle KQN = 142°$, how many degrees are there in $\angle r$? In $\angle s$? (Fig. p. 29.)

3. If $EB \perp AC$ and $\angle x = \angle w$, show that EB bisects $\angle FBD$.

4. If $\angle x$ and r are supplementary, why does $\angle r = \angle w$? $\angle p = \angle w$?



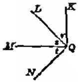
Ex. 1.

Some Geometric Forms in Nature

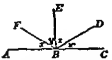
5. If $\angle p = \angle w$, show that $\angle r = \angle w$; that $\angle s$ is the supplement of $\angle w$.

6. If $\angle r = \angle w$, show that $\angle p = \angle y$.

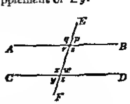
7. If $\angle r = \angle w$, show that $\angle q$ is the supplement of $\angle y$.



Ex. 2



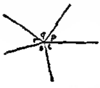
Ex. 3

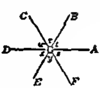


Exs. 4, 5, 6, 7.

8. If $\angle p = 70°$, $\angle q = 80°$, $\angle r = 50°$ and $\angle s = 90°$, find $\angle t$.

9. If $\angle$s p, q, r, s, and t are in the ratio $1:2:3:4:5$, find the number of degrees in each. (NOTE: $x + 2x + 3x + 4x + 5x = 360°$.)


Exs. 8, 9.



Exs. 10, 11, 12, 13.

OPTIONAL EXERCISES

10. If $\angle x$ is the complement of $\angle z$, how many degrees are there in $\angle r$?

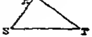
11. If $\angle t = 70°$ and $\angle v = 50°$, how many degrees are there in $\angle x$? In $\angle DGF$? In $\angle AGE$?

12. If $\angle y$ is a rt. angle, what is the sum of $\angle t$ and $\angle w$?

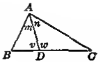
13. If $\angle t = \angle v = \angle w$, find the number of degrees in $\angle DGF$.

14. If $\angle CBF = 153°$, $\angle EBA = 101°$ and ABC is a straight line, find the number of degrees in $\angle EBF$. (See figure of Ex. 3.)

15. If $\angle x = \angle T$ and $\angle z = \angle S$, how many degrees are there in the sum of the angles of $\triangle RST$?



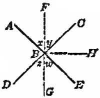
16. If $\angle v = \angle n + \angle C$ and $\angle w = \angle m + \angle B$, how many degrees are there in $\angle A + \angle B + \angle C$?



Honor Work

17. What angle is formed by the bisectors of two supplementary adjacent angles?

18. If the bisectors FB and HB of $\angle s\ ABC$ and EBC are perpendicular to each other, why is ABE a straight line?



19. If FG bisects $\angle ABC$, prove that it bisects its vertical $\angle DBE$ also

20. If $\angle x = \angle y$, $\angle z = \angle w$, and AE and CD are straight lines, prove that FBG is a straight line.

21. If FB bisects $\angle ABC$ and BH bisects $\angle CBE$, prove that $BH \perp FB$.

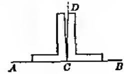
Applied Problems

22. The draughtsman uses two right triangles, one having acute angles of 30° and 60°, and the other having two 45° angles. Name nine different sized angles which he can draw, using either one triangle alone or both together.

23. In physics, it is found that when a ray of light AB strikes a plane mirror EF, it is reflected in a direction BC so that $\angle y = \angle z$, where BD is perpendicular to the mirror. Show that the ray of light also makes equal angles with the mirror, that is, that $\angle x = \angle w$.

24. Paul Strong bought a carpenter's square at a sale, but he suspects that the right angle is not true. To test it, he places one

edge on a straight line AB, and draws a line CD along the other edge. He then turns the square over, as shown in the diagram, and notices whether the edges fit the lines drawn. Prove that if the square has a true right angle, it will so fit.



25. Dorothy wished to draw a perpendicular to a line but had no instrument with which to do it, so she folded an irregular piece of paper twice and obtained a right angle along which she could draw. Can you do it too? Explain why this gives you a right angle.

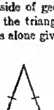
Polygons; Triangles

38. A **polygon** is a figure formed by straight lines which enclose a portion of the plane. Its **perimeter** is the sum of the lines.

39. A **triangle** ($\triangle$) is a polygon having three sides.

The triangle is important both in geometry and in its applications. In geometry, any polygon can be cut into triangles, and in that way proofs can be simplified. Outside of geometry, in construction work, such as bridge-building, the triangle is important because it is the only polygon whose sides alone give rigidity.

40. An **isosceles triangle** is a triangle having at least two of its sides equal. The two equal sides are the **legs**, and the angle formed by them is the **vertex angle**. The third side is the **base**, and the angles adjoining it are the **base angles**.



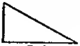
Isosceles Triangle

Equilateral Triangle

41. An **equilateral triangle** is a triangle having all three sides equal.

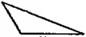
An *equilateral triangle* is a special kind of isosceles triangle

42. A right triangle is a triangle one of whose angles is a right angle. The side opposite the right angle is the **hypotenuse**, and the other two sides are the **legs**.



Right

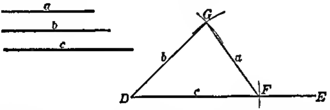
An *obtuse triangle* has one obtuse angle. An *acute triangle* has three acute angles. An *equiangular triangle* is one in which all of the angles are equal.



Obtuse

Acute

How to construct a triangle when the three sides are known.



Suppose the line segments a, b and c are to be the sides of your triangle.

1. Draw a straight line DE of indefinite length.

2. With D as center and c as radius, cut DE at F.

3. With D as center and b as radius, construct an arc.

4. With F as center and a as radius, cut the last arc at G.

5. Draw DG and FG.
DFG is the required triangle.

Construction Exercises]

1. Construct a triangle whose sides are 2 in., 3 in., and 3½ in. With your protractor measure each angle of your triangle. Are

there any equal angles? How many degrees are there in the sum of all three angles?

2. Construct an equilateral triangle each of whose sides is $2\frac{1}{2}$ in. Are there any equal angles? What is the sum of all three angles?

3. Construct an isosceles triangle whose base is 2 in. and whose legs are each 3 in. Are any of the angles equal? Find their sum.

4. Construct an equilateral triangle whose base is a given line segment a.

5. Construct an isosceles triangle given the base a and a leg b.

6. (a) Construct a $\triangle ABC$ having $AB=3$ in., $AC=1$ in. and $BC=3\frac{1}{2}$ in.

(b) Then construct the bisector of $\angle A$. Is this line perpendicular to BC? Does it bisect BC?

7. Construct an isosceles triangle and bisect its vertex angle. Is this line perpendicular to the base? Does it bisect the base? Does this appear to be true of isosceles triangles only?

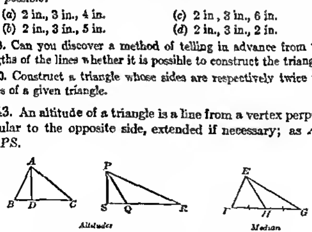
8. With the following lengths as sides, construct triangles when it is possible:

(a) 2 in., 3 in., 4 in.
(b) 2 in., 3 in., 5 in.
(c) 2 in., 3 in., 6 in.
(d) 2 in., 3 in., 2 in.

9. Can you discover a method of telling in advance from the lengths of the lines whether it is possible to construct the triangle?

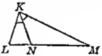
10. Construct a triangle whose sides are respectively twice the sides of a given triangle.

43. An altitude of a triangle is a line from a vertex perpendicular to the opposite side, extended if necessary; as AD or PS.



44. A median of a triangle is a line from a vertex to the middle point of the opposite side; as EH above.

45. An angle bisector of a triangle is a line bisecting an angle of a triangle and extending to the opposite side; as KN.

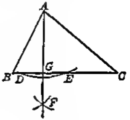


A triangle has three altitudes, three medians, and three angle bisectors.

How to construct an altitude of a given triangle.

To construct the altitude from A to BC:

1. Take A as center, and with a convenient radius cut BC at D and E.



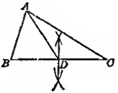
2. With D as center and a convenient radius draw an arc.

3. With E as center and the same radius, draw an arc cutting the last arc at F.

4. Draw AF cutting BC at G.
AG is the required altitude.

Note: If $\angle B$ or $\angle C$ were obtuse, it would be necessary to extend the side BC. Why?

How to construct a median of a given triangle.



Since the median from A must pass through the middle point of BC, we must first bisect BC at D. Then AD is the required median.

Construction Exercises

1. Draw an acute triangle. Construct all three altitudes. Are all three inside the triangle? Do they meet in a point?

2. Draw an obtuse triangle. Construct all three altitudes. Are all three inside the triangle? Do they meet in a point?

3. Draw a right triangle. Construct all three altitudes. What do you notice about the position of these altitudes?

4. Draw any triangle. Construct the perpendicular bisectors of all three sides. Do they meet in a point?

5. Draw an acute triangle ABC and construct the median to AC.

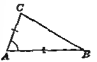
6. Draw an obtuse triangle and construct all three medians. Do they meet in a point?

7. Draw a triangle and construct all three angle bisectors. Do they meet in a point?

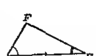
8. (a) Construct a $\triangle ABC$ having $AB=1$ in., $AC=3\frac{1}{2}$ in., and $BC=3$ in.
   (b) From A construct the altitude, median and angle bisector.
   (c) Are they three separate lines? Which is between the other two?

9. (a) Construct an equilateral triangle ABC having each side 3 in.
   (b) From A construct the altitude, median, and angle bisector.
   (c) Are they three separate lines? If not, which of them coincide?

10. Construct a triangle whose sides are $1\frac{1}{2}$ times the sides of a given triangle.



46. When two sides of a triangle are mentioned, the angle formed by these two sides is called the **included angle.** In the figure, $\angle A$ is included by AB and AC.

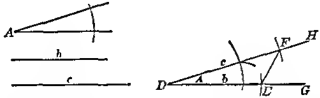


47. When two angles of a triangle are mentioned, the side joining their vertices is called the **included side.** *DE is included by $\angle D$ and $\angle E$.*

In $\triangle ABC$ what angle is included by AC and BC? By AB and BC? What side is included by $\angle A$ and $\angle B$? By $\angle B$ and $\angle C$? By $\angle A$ and $\angle C$?

How to construct a triangle when two sides and their included angle are given.

Given $\angle A$ and line segments **b** and **c.**



1. On a line DG using D as center and b as radius, cut DG at E.
2. Construct an angle at $D = \angle A$.
3. With D as center and c as radius, cut DH at F.
4. Draw EF.

$\triangle DEF$ is the required triangle.

Construction Exercises

1. Using your protractor, draw an angle of 50°. Now with your compasses measure off a length of 1 in. on one side and $1\frac{1}{2}$ in. on the other. Complete the triangle by drawing a line joining the ends of the segments marked off.

2. Draw an angle of 70°, and two line segments, one 2 in. long and the other 3 in. long. Now on another line construct with your compasses a triangle having these lines and this angle as two sides and their included angle.

3. Draw an angle of 40°. Construct an isosceles triangle having its vertex angle equal to this angle, and having a leg equal to 2 in. Measure the three angles with your protractor and find their sum.

4. Draw two angles, one about 70° and the other 65°, and draw a line 3 in. long. Now construct a triangle on another line having two angles equal to these angles and the included side equal to this line segment.

5. Construct a right angle and an angle of 45°. Draw a line

segment $2\frac{1}{4}$ in. long. Now construct a triangle having two angles and their included side equal to these parts.

6. Construct a right angle. Then construct an isosceles triangle having its vertex angle equal to this angle, and each leg equal to 2 in.

7. Construct a triangle whose base is 3 in., and whose base angles are each 45°.

6. Construct $\triangle ABC$ making $BC = 2$ in., $\angle B = 90°$ and $\angle C = 45°$.

9. Construct a right triangle whose legs are 2 in. and 3 in.

10. Draw an angle of 130°. Now construct a triangle having an angle equal to this angle and the including sides 2 in. and $1\frac{1}{2}$ in.

11. (*a*) Draw a triangle. Construct another triangle having 2 sides and the included angle equal to 2 sides and the included angle of this triangle. Do the triangles appear to be congruent?

(*b*) Cut out one of them and see if it will fit the other. Are they congruent?

12. Try the same experiment beginning with a triangle having a different shape and size. Are two triangles always congruent when two sides and the included angle of one equal two sides and the included angle of the other?

13. In two triangles ABC and $A'B'C'$, if you knew that $\angle A = A'$, that AB is the same length as $A'B'$, and AC the same length as $A'C'$, would you know that $\triangle ABC \cong \triangle A'B'C'$?

Two triangles are congruent, if two sides and the included angle of one equal respectively two sides and the included angle of the other. (s.a.s.)

14. Draw a triangle. Construct another triangle having two angles and the included side equal to two angles and the included side of this triangle. Do these triangles appear to be congruent? Test by cutting out one of them and fitting on the other.

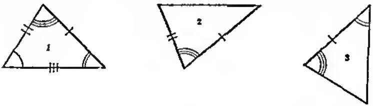
15. Try the same experiment beginning with a different shaped triangle. Are two triangles always congruent when two angles and the included side of one equal two angles and the included side of the other?

Two triangles are **congruent** if **two** angles and the included side of one equal **respectively two** angles and the included side of the other. (a.s.a.)

16. Among the following triangles select two pairs which are congruent and tell how you know.

17. In these figures sides and angles are equal when they are marked alike. Find a triangle that is congruent to triangle 1; to triangle 2. Explain how you know that the triangles selected are the correct ones.

18. In these figures, is triangle 1 congruent to triangle 2? How do you know? Is triangle 1 congruent to triangle 3? How do you know?



48. A quadrilateral is a **four-sided** polygon. A quadrilateral, whose sides are all equal, is a **rhombus**, and if the angles are right angles is also a **square**. A **diagonal** is a line joining opposite vertices.

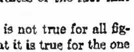
49. Need of a proof. A pupil often jumps to the conclusion that a statement is true simply because he can see that it is true from the figure before him. He does not yet understand that we are not trying to prove anything about that particular figure. We are proving a statement for all figures to which that hypothesis applies, and we use the figure on the paper or blackboard as a convenience only.

Suppose, for example, that one child tries to prove to another that all birds can fly. He says "I have seen them fly. The crow, the robin, the swallow, the eagle, and the sparrow all fly." "But," the other remarks, "there may be a bird which you have not seen which cannot fly." And he is right. The ostrich cannot fly. Therefore the statement that all birds can fly is not true, regardless of the fact that all the birds that he has seen can fly.

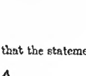
Similarly, in geometry, a statement is not true for all figures simply because a pupil can see that it is true for the one he has drawn. It is quite possible that, in another figure, he could see equally well that the statement was not true at all. Consequently, he must never accept a statement simply because it looks as if it were true.

Exercises

In each of the following figures, we can see that the statement is true for the figure drawn. Try to draw another figure for which it is not true. If you do not succeed in doing this, are you sure that no one else could?



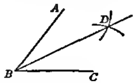
1. An angle outside a triangle, made by extending one of the sides, is larger than any one of the angles of the triangle.



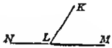
2. A perpendicular to a line bisects the line.

3. The bisector of a straight angle bisects the straight line.

4. If, with the same radius, arcs are drawn from A and C as centers, meeting at D, BD bisects angle ABC.



5. The supplement of an angle is always obtuse.



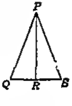
6. Adjacent angles are supplementary.

7. The bisector of an angle of a triangle bisects the opposite side and forms right angles with it.

8. An isosceles triangle has *three acute angles.*

9. From the vertex P of a triangle PQS, we can construct a line PR perpendicular to QS and bisecting QS.

10. A median to the base of a triangle is perpendicular to the base.



True-False Test (*10 min.*)

Write the numbers 1 to 10 on your answer paper. If the statement is true, write T after its number, if false, write F

1. The diameter of a circle is a chord.
2. An acute angle is an angle less than a right angle.
3. Adjacent angles are two angles whose sum is a straight angle.
4. Supplementary angles are equal.
5. An axiom is a statement to be proved.
6. The sum of the angles about a point is two straight angles.
7. A polygon is a figure having five sides.
8. The angle formed by two sides of a triangle is called their included angle.
9. A median of a triangle is a line from a vertex to the opposite side.
10. An equilateral quadrilateral is a rhombus.

Completion Test (*10 min.*)

Write the numbers of the questions on your answer paper, and after each the one word that is omitted.

1. Any polygon that has just four sides is called a . . .

2. The difference between the supplement and the complement of an angle is always . . . degrees.

3. In a triangle the line from a vertex perpendicular to the opposite side, extended if necessary, is called the . . .

4. The bisectors of two supplementary adjacent angles form an angle of . . . degrees.

5. A triangle is . . . if two of its altitudes fall outside the triangle.

Reasoning Test (*10 min.*)

In the following exercises you are given certain angles and certain lines equal. Select another pair of lines or angles, one in each triangle, which are equal as a result of these, and give a reason for your selection.

1. $BG = EH$ and $GC = HF$.
2. $BC = EF$ and $BG = EH$.
3. $BC = EF$ and AG and DH are medians.
4. $\angle A = \angle D$ and AG and DH are angle bisectors.
5. AG and DH are altitudes.

Matching Test (*10 min.*)

Write the numbers 1 to 10 in a column. After each write the letter of the phrase which is a correct definition of the word following the number.

1. Adjacent angles	*a.* The line from a vertex of a triangle to the middle point of the opposite side.

2. Polygon	*b.* Two angles whose sum is 180°.
3. Altitude	*c.* The side of a right triangle opposite the right angle.
4. Median	*d.* A statement to be proved.
5. Isosceles triangle	*e.* Two angles which have the same vertex and a common side between them.
6. Acute angle	*f.* An angle smaller than a right angle.
7. Supplementary angles	*g.* Two lines meeting at right angles.
8. Theorem	*h.* A figure formed by straight lines which enclose a portion of the plane.
9. Hypotenuse	*i.* A triangle having at least 2 sides equal.
10. Perpendicular lines	*j.* A line from a vertex of a triangle perpendicular to the opposite side.

DIFFERENTIATION OF PROPOSITIONS

Propositions on the fundamental list of the National Committee on Mathematical Requirements are indicated by boldface italic type; for example:

66. An angle equal to a given angle can be constructed at a given point on a given straight line.

Propositions whose proof is required by the College Entrance Examination Board are preceded by a star (*); thus:

**50. Two triangles are congruent if two sides and the included angle of one equal respectively two sides and the included angle of the other.*

TO THE PUPIL

You will notice that in some propositions part of the hypothesis is enclosed in brackets while another part is left outside of them. This is done to help you distinguish between those facts which the figure itself tells you and the more essential but less obvious ones which you will need as reasons in the proof. For example, on the next page, you can see that ABC is a triangle but not that AB is *exactly* equal to $A'B'$. You should be particularly careful to know that part not enclosed in the brackets before you try to prove the proposition.

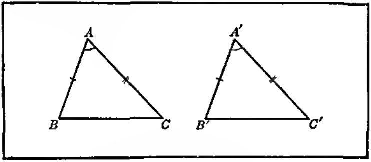
In propositions in which either all or none of the hypothesis is observable from the figure, the brackets are not used.

BOOK ONE

RECTILINEAR FIGURES

Proposition 1

★ **50.** *Two triangles are congruent, if two sides and the included angle of one equal respectively two sides and the included angle of the other.* (s.a.s.)



Given: [$\triangle ABC$ and $A'B'C'$]; $AB = A'B'$, $AC = A'C'$, and $\angle A = \angle A'$.

To prove: $\triangle ABC \cong \triangle A'B'C'$.

Proof:

Statements	Reasons
1. Place $\triangle ABC$ on $\triangle A'B'C'$ so that point B is on point B', and BA falls on $B'A'$.	1. A geometric figure can be moved without change of size or shape.
2. Point A is on point A'.	2. $AB = A'B'$ by hyp.
3. AC is on $A'C'$.	3. $\angle A = \angle A'$ by hyp.
4. Point C is on point C'.	4. $AC = A'C'$ by hyp.
5. BC is on $B'C'$.	5. Only one st. line . . . (§ 4).
6. $\triangle ABC \cong \triangle A'B'C'$	6. Figures which can be made to coincide are congruent.

51. Arrangement of proof. On page 43 is given a model proof. Note that it is arranged in the order suggested below. This order should always be followed by the pupil in arranging his written work.

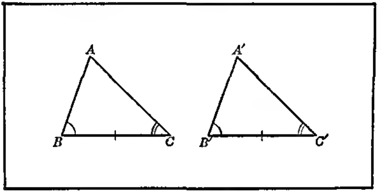
1. At the top of the paper, write out in words the proposition to be proved.
2. Draw a neat figure with pencil and ruler. The figure should not be too small. Preferably, no line should be less than one-and-one-half inches in length.
3. Mark the given parts on the figure.
4. Write the hypothesis in terms of the figure, that is, say, "Given $AB = CD$" and not "Given two equal lines."
5. State in terms of the figure what is to be proved.
6. Draw a straight line down through the middle of the page. On the left of this line, write and number the statements. On the right of the line, opposite each statement, write its reason, numbered to correspond with the statement.
7. When construction is necessary, it should be explained before the proof, with a reason for each construction.
8. In the proof, a reason must be given for every statement. No reason is allowable, no matter how evident, unless it has previously been accepted as true in the geometry. Such reasons are: axioms, postulates, definitions, propositions already proved, identity, construction, and the hypothesis.

Investigation Problem.—How many parts must be known to be equal in order that you can prove triangles congruent by Proposition 1? Can you select a different combination of parts which, if known, would enable you to prove triangles congruent? Try to draw triangles (1) that are not congruent but have all three angles of one equal to angles of the other; (2) that have two angles and the included side of one equal to two angles and the included side of the other.

NOTE: Unless needed to meet the requirements of an outside examination, the proofs of Propositions 1 and 2 should be omitted.

PROPOSITION 2

★ 52. *Two triangles are congruent, if two angles and the included side of one equal respectively two angles and the included side of the other.* (a.s.a.)



Given: [$\triangle ABC$ and $A'B'C'$]; $\angle B = \angle B'$, $\angle C = \angle C'$, and $BC = B'C'$.

To prove: $\triangle ABC \cong \triangle A'B'C'$.

Proof:

STATEMENTS	REASONS
1. Place $\triangle ABC$ on $\triangle A'B'C'$ so that point B is on point B', and BA takes the direction $B'A'$.	1. A geometric figure can be moved without change of size or shape.
2. Then BC takes the direction of $B'C'$.	2. $\angle B = \angle B'$ by hyp.
3. Point C is on point C'.	3. $BC = B'C'$ by hyp.
4. CA takes the direction $C'A'$.	4. $\angle C = \angle C'$ by hyp.
5. Point A is on point A'.	5. Two st. lines can meet in only one point.
6. $\triangle ABC \cong \triangle A'B'C'$.	6. Figures which can be made to coincide are congruent.

THE TRIANGLE IN ENGINEERING

The triangle has one quality, not possessed by any other polygon, that makes it particularly valuable in the construction of bridges, buildings, or other structures where strength is desired. That is its rigidity. Other polygons can be made rigid by constructing rigid joints. But the joint is the weakest point in construction, and if a joint gives, the polygon will change its shape. In the triangle, however, rigidity depends on the sides alone. Unless a side bends, the triangle cannot change its shape, no matter how flexible the joints may otherwise be. They may even be hinged or held by a single rivet.

Examine the figure on the opposite page and try to determine in how many different ways the triangle is used to strengthen this frame. Can you find triangles in a vertical plane? Why are they there? Are there triangles in a horizontal plane? In what way do they strengthen the frame?

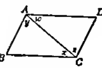
Another quality possessed by the triangle alone is of great value to the surveyor. When two sides and their included angle, or two angles and their included side, are known, the lengths of the other sides and the sizes of the other angles can be computed. Now it is often much easier to look through the telescope of a transit at a distant point and measure an angle than to measure the distance to that point. So the triangle saves the surveyor a great amount of work.

Photo by Paul J. Woolf

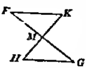
Class Exercises

1. If $AD = BC$ and $\angle w = \angle x$, prove that $\triangle ABC \cong \triangle ACD$.

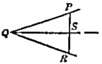
2. If $\angle w = \angle x$ and $\angle y = \angle z$, prove that $\triangle ABC \cong \triangle ACD$.



Exs. 1, 2.



Ex. 3.



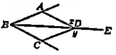
Ex. 4.

3. If the lines FG and KH bisect each other at M, prove $\triangle FKM \cong \triangle GHM$.

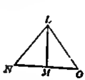
4. If QS bisects $\angle PQR$ and $PR \perp QS$, prove $\triangle PQS \cong \triangle RQS$.

5. If $\angle ABC$ is bisected by BE and $AB = BC$, prove $\triangle ABD \cong \triangle CBD$.

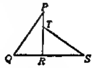
6. If $\angle ABC$ is bisected by BE and $\angle x = \angle y$, prove $\triangle ABD \cong \triangle CBD$.



Exs. 5, 6.



Ex. 7.



Exs. 8, 9.

7. If $LM \perp NO$ and bisects NO, prove $\triangle LMN \cong \triangle LMO$.

8. If $PR \perp QS$, $QR = TR$ and $PR = RS$, prove $\triangle PQR \cong \triangle STR$.

9. If $PR \perp QS$, $QR = TR$, $\angle Q = 70°$ and $\angle PTS = 110°$, prove $\triangle PQR \cong \triangle STR$.

10. The bisector of the vertex angle of an isosceles triangle divides the figure into two congruent triangles.

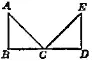
11. Two right triangles are congruent if the legs of one equal respectively the legs of the other.

12. Two right triangles are congruent if a leg and the adjoining acute angle of one equal respectively a leg and the adjoining acute angle of the other.

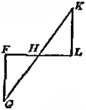
13. Two isosceles triangles are congruent if a leg and the vertex angle of one equal a leg and the vertex angle of the other.

14. If C is the middle point of BD, $AB=ED$, $AB\perp BD$ and $ED\perp BD$, prove $\triangle ABC\cong\triangle EDC$.

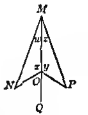
15. If $\angle E=40°$, $ED\perp BD$, $EC\perp AC$, $AB\perp BD$, $ED=BC$ and $\angle ECD=50°$, prove $\triangle ABC\cong\triangle CDE$.



Exs. 14, 15.



Ex. 16.



Exs. 17–21.

16. If $FG\perp FL$ and $KL\perp FL$ and KG passes through the middle point of FL, prove that $\triangle FHG\cong\triangle LHK$.

17. If $\angle x=\angle y$ and $NO=OP$, prove $\triangle MNO\cong\triangle MPO$.

18. If MQ bisects $\angle NMP$ and $\angle x=\angle y$, prove $\triangle MNO\cong\triangle MPO$.

19. If MQ bisects $\angle NOP$ and $NO=OP$, prove $\triangle MNO\cong\triangle MPO$.

20. If MQ bisects both $\angle NMP$ and $\angle NOP$, prove $\triangle MNO\cong\triangle MPO$.

21. If MQ bisects $\angle NMP$ and $\angle w+x=z+y$, prove $\triangle MNO\cong\triangle MPO$.

53. Method of attack. *To prove lines equal or angles equal,* choose a pair of triangles of which they are the corresponding parts. Prove these triangles congruent by Proposition 1 or Proposition 2. Then use:

54. *The corresponding sides and angles of congruent triangles are equal.*

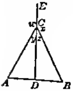
22. If the bisector of an angle of a triangle is perpendicular to the opposite side, the triangle is isosceles.

23. If two opposite angles of a quadrilateral are bisected by a diagonal connecting their vertices, the quadrilateral has two pairs of equal sides.

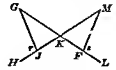
24. If the middle point of a side of a square is joined to the two vertices of the opposite side, the lines so drawn are equal.

25. If CD is the perpendicular bisector of AB, then $AC=BC$.

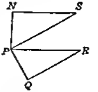
26. If $ED \perp AB$ and $\angle w = \angle z$, then $\angle A = \angle B$.



Exs. 25, 26.



Exs. 27, 28.



Ex. 29.

27. If HM and GL intersect at K so that $JK=KF$ and $GK=KM$, then $\angle G = \angle M$.

28. If HM and GL intersect at K so that $JK=KF$ and $\angle r = \angle s$, then $GK=KM$.

29. If NS and PR are perpendicular to NP, PS and QR perpendicular to PQ and $NP=PQ$, prove that $PS=PR$.

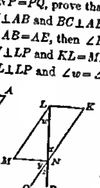
30. If $AD=AC$, $DE \perp AB$ and $BC \perp AE$, then $DE=BC$.

31. If $AD=AC$ and $AB=AE$, then $\angle B = \angle E$.

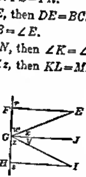
32. If $KL \perp LP$, $MN \perp LP$ and $KL=MN$, then $\angle K = \angle M$.

33. If $MN \perp LP$, $KL \perp LP$ and $\angle w = \angle z$, then $KL=MN$.

Exs. 30, 31.



Exs. 32–35.



Exs. 36–40.

34. If $LM=NK$, $MN \perp LP$ and $\angle w$ is complementary to $\angle y$, then $KL=MN$.

35. If $\angle w = \angle z$, $LM=NK$ and $KL \perp LP$, then $MN \perp LP$.

36. If G is the middle point of FH, $EG=GI$, $\angle x = \angle y$ and $GJ \perp FH$, then $EF=IH$.

37. If FH is bisected at G, $\angle r = \angle s$ and $\angle w = \angle z$, then $\angle E = \angle I$.

38. If $EG=GI$, $\angle E=\angle I$, $\angle x=\angle y$ and $GJ \perp FH$, then G bisects FH.

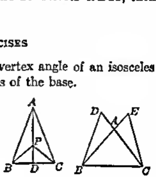
39. If G is the middle point of FH, $GJ \perp FH$, GJ bisects $\angle EGI$, $GE=GI$, and $EF \perp FH$, then $IH \perp FH$.

40. If $\angle r=\angle s$, $EF=IH$, $FG=GH$ and GJ bisects $\angle EGI$, then $GJ \perp FH$.

Optional Exercises

41. Any point in the bisector of the vertex angle of an isosceles triangle is equally distant from the ends of the base.

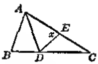
42. If the equal sides of an isosceles triangle ABC are produced through the vertex A equal lengths to D and E, then DB equals EC.



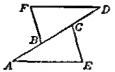
43. If, in $\triangle ABC$, AD bisects $\angle A$ and $AB=AE$, prove that $\angle B=\angle x$.

44. If $AB=CD$, $AE=FD$, and $\angle A=\angle D$, then $CE=FB$.

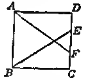
45. If $AB=CD$, $BF=CE$, and $\angle ABF=\angle DCE$, then $\angle A= \angle D$.



Ex. 43.



Exs. 44, 45, 46.



Ex. 47.

46. If $AB=CD$, $BF \perp AD$, $CE \perp AD$ and $\angle A=\angle D$, then $\angle F=\angle E$.

47. If $ABCD$ is a square and $GF=DE$, then $AF=BE$.

48. If, in $\triangle ABC$, D is the middle point of AC, and BD is extended its own length to E, then $\angle A=\angle ACE$.

49. In congruent triangles, corresponding medians are equal.

50. In congruent triangles, corresponding angle bisectors are equal.

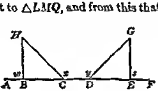
51. If $BD=CE$, $HB\perp AF$, $GE\perp AF$, and $\angle x=\angle y$, then $HC=DG$.

52. If $BD=CE$, $\angle w=\angle z$, and $\angle x=\angle y$, then $\angle H=\angle G$.

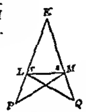
Honor Work

53. If the legs KL and KM of an isosceles triangle KLM are extended equal lengths to P and Q, then $PM=LQ$.

54. Using the same hypothesis, prove that $\triangle LPM$ is congruent to $\triangle LMQ$, and from this that $\angle r=\angle s$.



Exs. 51, 52.



Exs. 53, 54.

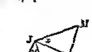
55. If two rods AB and BC are hinged at B, can the distance between the ends A and C be changed? If now the size of $\angle B$ is fixed by another pin so that it cannot become larger or smaller, what can you say about the distance from A to C? Give a reason.

56. If $AB=AC$, and BD and CF are medians, prove that $\triangle ABD\cong\triangle ACF$; then that $\triangle FBC\cong\triangle DCB$.

Ex. 56.

Ex. 57.

Ex. 58.



Ex. 59.

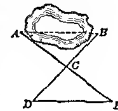
57. If $AB=AC$, prove that their perpendicular bisectors DE and FG are equal; also that $\triangle DGH\cong\triangle FEH$.

58. If $AB=AC$, $AD=AF$, $AD\perp AB$, and $AF\perp AC$, prove that $\triangle ADC\cong\triangle ABF$.

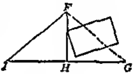
59. If $JK=JL$, $JM=JN$, and $\angle x=\angle y$, prove that $\triangle JKN\cong\triangle JLM$.

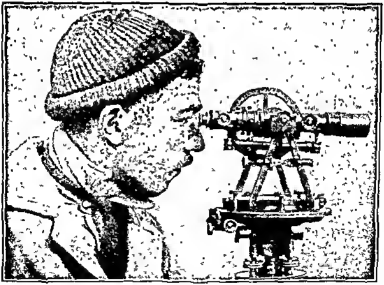
Applied Problems

60. To find the distance across a pond from A to B, Robert Young set up a pole at C and measured a distance CD equal to BC, and a distance CE equal to AC. Show that he can find the required distance by measuring DE.



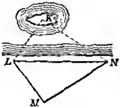
61. Engineers for the Lively Traffic R.R. Co. are surveying a right of way through John Greer's farm. At one place their work is obstructed by a barn. They want to know the distance from F to G. They lay off any line HG and make FH perpendicular to it. HI is then taken equal to HG. Show that FI is the same length as the distance from G to F.



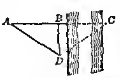


Surveyor Using a Transit

62. To measure the distance from a point L on the shore to an island K, a surveyor found that $\angle KLN$ equaled 40° and $\angle KNL$ equaled 35°. He then laid off a triangle LMN with $\angle NLM = 40°$ and $\angle LNM = 35°$. Show what line he had to measure to find the distance from L to K.



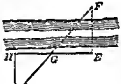
Ex. 62.



Ex. 63.

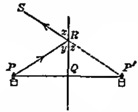
63. To find the distance from B across a river to a point C, BD is drawn perpendicular to BC. At the point D, $\angle BDA$ is constructed equal to $\angle BDC$. Prove that BA equals BC.

64. A man can find the distance EF across a river as follows. He measures EH at right angles with EF, and sets a stake at the middle point G. Then he measures the distance at right angles from H until he arrives at a point I at which he is in line with G and F. Show the EF equals the distance he has measured.



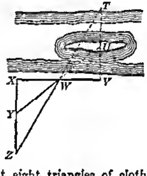
65. Two boys, Henry and James, find a distance KL as follows. Henry walks at right angles to KL any distance KM and measures the $\angle KML$ with his protractor. He calls back to James, who is at K, that the angle is 25°. The latter then measures an angle of 25°, $\angle MKN$, and travels along KN until Henry, who is at M, observes that he is at right angles with MK. They decide that MN equals KL. Prove that they are right.

66. QR is a mirror, and light from P is reflected at R to S so that $\angle x = \angle y$. The image P' of P appears to be on SR produced through the mirror, and also on a line PP' perpendicular to the mirror. Prove that PQ and QP' are equal in length.



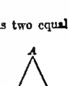
67. When an object, such as candle P, is reflected in a mirror QR, the image P' is as far from the mirror as the object is, and is on the perpendicular from the object to the mirror produced through the mirror. If SRP' is a straight line, prove that $\angle x$ equals $\angle y$.

68. To find the distance between two points T and U, both inaccessible, take XV at right angles to the line TU extended, and set up a post at its middle point W. Then walk along XZ, perpendicular to XV, and set stakes at Y and Z in line with UW and TW. Prove that YZ is the required distance.



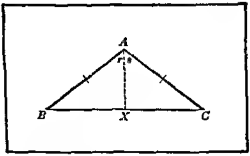
69. Mr. Wilson asked his wife to cut eight triangles of cloth to cover his umbrella. He told her that the ribs were each 26 in. long, and that when the umbrella was open, they met at angles of 45°. Mrs. Wilson insists that she cannot cut the triangles the correct size unless she also knows how far apart the ribs are at their outer ends. This extra information Mr. Wilson says is unnecessary. Which is right? Can she cut the triangles from the data given her? Explain your answer.

Investigation Problem. $\triangle ABC$ is isosceles. It has two equal sides AB and AC. Has it other equal parts? Can you prove them equal? What is the principal method of proving angles equal? If there is only one triangle in the figure, how can you do this? Would it help to have a line bisecting $\angle A$? If you cannot prove your conclusion without help, read Proposition 3.



Proposition 3

* **55.** *The base angles of an isosceles triangle are equal.*



Given: [$\triangle ABC$]; $AB = AC$.
To prove: $\angle B = \angle C$.

Proof: Statements	Reasons
1. Let AX bisect $\angle BAC$.	1. Every angle has a bisector.
2. In $\triangle ABX$ and ACX, $AB = AC$.	2. Hyp.
3. $AX = AX$.	3. Iden.
4. $\angle r = \angle s$.	4. Const.
5. $\triangle ABX \cong \triangle ACX$.	5. Two $\triangle$ are $\cong$, if two sides and the included angle of one equal ... (§ 50).
6. $\angle B = \angle C$.	6. Corr. $\angle$s of $\cong$ $\triangle$ are equal.

56. Corollary. *An equilateral triangle is equiangular.*

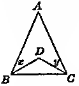
Class Exercises

1. In an isosceles triangle, the exterior angles made by producing the base are equal.

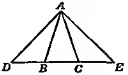
2. Two isosceles triangles are congruent, if the base and a base angle of one triangle equal the base and a base angle of the other.

3. If ABC and DBC are isosceles triangles on the same base BC, prove that $\angle x = \angle y$.

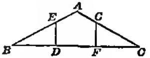
4. The lines from the vertex of an isosceles triangle to the trisection points of the base are equal. [To trisect is to divide into 3 *equal parts.*]



Ex. 3.



Exs. 5, 6.



Exs. 8, 9.

5. If the base BC of isosceles $\triangle ABC$ is extended so that $BD = CE$, then $\angle D = \angle E$.

6. If, in $\triangle ABC$, $AB = AC$, and $\angle DAB = \angle EAC$, then $BD = CE$.

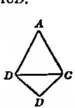
7. The two straight lines which join the middle points of the legs of an isosceles triangle to the middle point of the base are equal.

8. If $\triangle ABC$ is isosceles, and perpendiculars to the base are drawn making $BD = FC$, then the perpendiculars are equal.

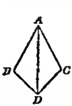
9. If, in isosceles $\triangle ABC$, E and G are the middle points of the legs, and D and F the trisection points of the base, prove that $DE = FG$.

10. The median to the base of an isosceles triangle bisects the vertex angle.

11. If $\triangle ABC$ and DBC are isosceles on the same base BC, then $\angle ABD = \angle ACD$.



Ex. 11.



Ex. 12.

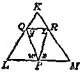
12. If, in quadrilateral $ABDC$, $AB = AC$, and $BD = DC$, then $\angle B = \angle C$, and the diagonal AD cuts the figure into two congruent triangles.

Optional Exercises

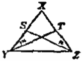
13. If $KL = KM$, P is the middle point of LM, and $\angle w = \angle x$, prove that $PQ = PR$.

14. Given the same hypothesis as in Ex. 13, prove that $\angle y = \angle z$.

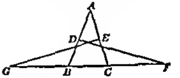
15. The angle bisectors of the base angles of an isosceles triangle are equal.



Exs 13, 14.



Ex. 17



Exs. 18, 19.

16. The medians to the legs of an isosceles triangle are equal.

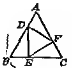
17. If $XY = XZ$ and $\angle m = \angle n$, prove that $YT = SZ$.

18. If $AB = AC$, $BD = CE$, $DF \perp AB$, and $GE \perp AC$, prove that $DF = GE$.

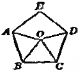
19. Perpendicular bisectors of the legs of an isosceles triangle terminating in the base, or the base produced, are equal.

20. The lines joining the middle points of the three sides of an equilateral triangle form another equilateral triangle

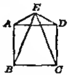
21. If $AB = BC = CA$ and $AD = BE = CF$, then $\triangle DEF$ is equilateral.



Exs. 20, 21.



Ex. 22.



Ex. 23.

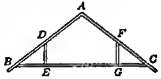
22. If $ABCDE$ is equilateral and equiangular, and $OB = OC$, then $OA = OD$.

23. If $ABCD$ is a square and $AE = ED$, then $EB = EC$.

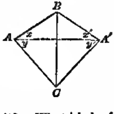
Applied Problems

24. Jasper King runs an antenna from the ridge of his house to the end of a pole which he fastens horizontally at the eaves, and he finds that he has just enough wire. However, Mr. King asks him to remove it from the front of the house and put it up in the rear. Jasper is a bit disappointed for he fears that the wire may not be long enough to reach. But his father argues that, since the two sides of the roof are equal, he will need exactly the same length of wire. Who was right? Prove your answer.

25. A carpenter wishes to support a roof, whose rafters AB and AC are equal, by upright pieces at E and G, equal distances from B and C, respectively. He drives a nail at D, suspends a weight, and measures DE. Prove that he can now cut both pieces this length without first measuring FG.

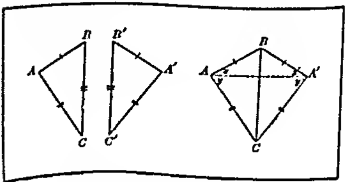


Investigation Problem. $\triangle A'BC$ has each of its three sides equal respectively to the corresponding sides of $\triangle ABC$. Do the triangles look congruent? Can you draw a triangle with sides equal to those of ABC that will not look congruent to it? Try it. Now try to see if your triangles can be made to coincide. What methods of proving triangles congruent do you know? To prove these triangles congruent, what other parts do you need equal? If $\angle A$ were equal to $\angle A'$, could you then prove it? What kind of triangle is ABA'? Does $\angle x$ equal $\angle x'$? Does $\angle y$ equal $\angle y'$? Now can you prove $\angle A$ equal to $\angle A'$? If you give up, look at Proposition 4.



Proposition 4

57. *Two triangles are congruent if the three sides of one equal the three sides of the other.* (s.s.s.)



Given: [$\triangle ABC$ and $A'B'C'$]; $AB = A'B'$, $AC = A'C'$, and $BC = B'C'$.

To prove: $\triangle ABC \cong \triangle A'B'C'$.

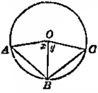
Proof: Statements	Reasons
1. Place $\triangle A'B'C'$ so that $B'C'$ coincides with its equal BC, and so that A' and A are on opposite sides of BC.	1. A geometric figure can be moved without change of size or shape.
2. Draw AA'.	2. A st. line can be drawn between any two points.
3. $AB = A'B$.	3. Hyp.
4. $\angle x = \angle x'$.	4. The base angles of an isos $\triangle$ are equal.
5. $AC = A'C$.	5. Hyp.
6. $\angle y = \angle y'$.	6. The base angles of an isos $\triangle$ are equal.
7. $\angle x + \angle y = \angle x' + \angle y'$, or $\angle BAC = \angle BA'C$.	7. If equals are added to equals, the results are equal.
8. $\triangle ABC \cong \triangle A'BC$ or $\triangle ABC \cong \triangle A'B'C'$.	8. Two $\triangle$ are $\cong$ if two sides and the included angle....

Class Exercises

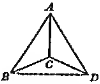
1. Construct a triangle, given the three sides.

2. The median to the base of an isosceles triangle divides it into two congruent triangles.

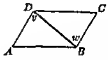
3. Two equilateral triangles are congruent, if a side of one equals a side of the other.



Ex. 4.



Ex. 5.



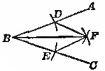
Ex. 6

4. In the circle whose center is O, if $AB = BC$, prove $\angle x = \angle y$.

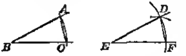
5. Using the accompanying figure, prove the theorem that two triangles are congruent if the three sides of one equal the three sides of the other.

6. If the opposite sides of the quadrilateral $ABCD$ are equal, then $\angle v = \angle w$.

7. If equal lengths BD and BE are taken on the sides of $\angle ABC$, and, from D and E, arcs are made with the same radius, intersecting at F, prove that BF bisects $\angle ABC$.



Ex. 7



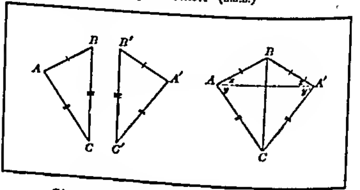
Ex. 8.

8. If AB, BC, DE, and EF are all measured with the same radius, and DF is measured with the radius AC, prove that $\angle E = \angle B$.

9. A square is divided by its diagonal into two congruent triangles.

Proposition 4

* 57. *Two triangles are congruent if the three sides of one equal the three sides of the other.* (s.s.s.)



Given: [$\triangle ABC$ and $A'B'C'$]; $AB = A'B'$, $AC = A'C'$, and $BC = B'C'$.

To prove: $\triangle ABC \cong \triangle A'B'C'$.

Proof:

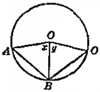
Statements	Reasons
1. Place $\triangle A'B'C'$ so that $B'C'$ coincides with its equal BC, and so that A' and A are on opposite sides of BC.	1. A geometric figure can be moved without change of size or shape.
2. Draw AA'.	2. A st. line can be drawn between any two points.
3. $AB = A'B$.	3. Hyp.
4. $\angle x = \angle x'$.	4. The base angles of an isos $\triangle$ are equal.
5. $AC = A'C$.	5. Hyp.
6. $\angle y = \angle y'$.	6. The base angles of an isos $\triangle$ are equal.
7. $\angle x + \angle y = \angle x' + \angle y'$, or $\angle BAC = \angle BA'C$.	7. If equals are added to equals, the results are equal.
8. $\triangle ABC \cong \triangle A'BC$ or $\triangle ABC \cong \triangle A'B'C'$.	8. Two $\triangle$ are $\cong$ if two sides and the included angle....

Class Exercises

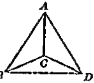
1. Construct a triangle, given the three sides.

2. The median to the base of an isosceles triangle divides it into two congruent triangles.

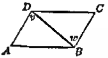
3. Two equilateral triangles are congruent, if a side of one equals a side of the other.



Ex. 4.



Ex. 5.



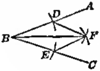
Ex. 6

4. In the circle whose center is O, if $AB = BC$, prove $\angle x = \angle y$.

5. Using the accompanying figure, prove the theorem that two triangles are congruent if the three sides of one equal the three sides of the other.

6. If the opposite sides of the quadrilateral $ABCD$ are equal, then $\angle v = \angle w$.

7. If equal lengths BD and BE are taken on the sides of $\angle ABC$, and, from D and E, arcs are made with the same radius, intersecting at F, prove that BF bisects $\angle ABC$.



Ex. 7

Ex. 8.

8. If AB, BC, DE, and EF are all measured with the same radius, and DF is measured with the radius AC, prove that $\angle E = \angle B$.

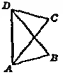
9. A square is divided by its diagonal into two congruent triangles.

10. If two circles whose centers are C and D intersect in A and B, then $\angle ACD = \angle BCD$.

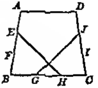
11. If $AB = CD$ and $AC = BD$, then $\angle C = \angle B$.

12. A diagonal of a rhombus bisects the angles.

Ex. 10.



Ex. 11



Ex. 13.

13. If, in the figure, $AB = CD$; E, F, G, H, I, and J are the trisection points of AB, BC, and CD respectively; and $EH = GJ$, prove that $\angle B = \angle C$.

14. If two isosceles triangles have the same base, the line joining their vertices bisects the vertex angles.

58. Analytic proof, or proof by analysis. Examine the following reasoning carefully to see if it really proves what it attempts to prove.

If AB equals BC, $\angle A$ equals $\angle C$ and $\angle x$ equals $\angle y$, prove that BF equals BG.

$BF = BG$ if $\triangle BFD \cong \triangle BGE$, for they are corresponding parts.

Now $\angle x = \angle y$ (hyp.) and $\angle v = \angle w$ (vertical $\angle$s are equal).

So $\triangle BFD \cong \triangle BGE$ if $BD = BE$, for they would then have two $\angle$s and the included side of one equal to two $\angle$s and the included side of the other.

But $BD = BE$ if $\triangle ABD \cong \triangle CBE$, for they are corresponding parts.

And $\triangle ABD \cong \triangle CBE$ for $\angle v = \angle w$ (vertical angles are

equal), $AB = BC$ (hyp.), and $\angle A = \angle C$ (hyp.). So the triangles have two angles and the included side of one equal to two angles and the included side of the other.

Therefore we have proved that $BF = BG$.

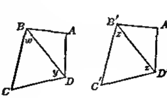
Does this proof convince you? Is there a reason for every step? Is there any mistake in the reasoning? How does it differ from the proof that you have had before? Could you work out this proof more easily than the other?

This new kind of proof is called **analytic proof**, or proof by analysis. Whereas the form of proof we have used before is called **synthetic proof**. Notice that the analytic proof begins with the statement to be proved, and works back to some known facts. The pupil should say to himself, "I can prove statement A if I can prove statement B. And I can prove statement B if I can prove statement C. But I know that statement C is true because of so and so. Therefore statement A is true."

Notice that in the above example, when we could not find enough parts to prove the first triangles congruent directly, we chose another pair of triangles whose corresponding parts were of use in the first pair, and proved the second pair of triangles congruent.

Illustration 1.

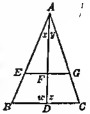
If in quadrilaterals $ABCD$ and $A'B'C'D'$, $\angle A = \angle A'$, $\angle w = \angle x$, $\angle y = \angle z$, $AB = A'B'$ and $AD = A'D'$, then $BC = B'C'$.



BC will equal $B'C'$ if $\triangle \ldots \cong \triangle \ldots$ Why? *But* $\angle w = \angle x$ *and* $\angle y = \angle z$. Why? Since we have 2 $\angle$s of one $= 2$ $\angle$s of the other, $\triangle \ldots$ will be $\cong \triangle \ldots$ if $\ldots = \ldots$ for they will have 2 $\angle$s and $\ldots$ BD will $= B'D'$ if $\triangle ABD \cong \triangle \ldots$. But $\triangle \ldots \cong \triangle \ldots$ because $\ldots = \ldots$, $\ldots = \ldots$ and $\ldots = \ldots$. Therefore $BC = B'C'$.

Illustration 2.

In $\triangle ABC$, altitude AD is the perpendicular bisector of EG. Then $AB = AC$.



AB will $= AC$ by corr. sides of $\cong \triangle$s if what triangles are congruent? But in these triangles, we have $AD = AD$. Why?

$\angle w = \angle z$. Why? Since we have 1 side and 1 $\angle$ of one triangle $=$ 1 side and corr. $\angle$ of the other, we can prove $\triangle ABD \cong \triangle ADC$ if we can get . . . or . . . , for 2 triangles are $\cong$ if . . . . But $\angle x$ will $= \angle y$ if $\triangle AEF \cong$ . . . . In these triangles we have . . . $=$ . . . , . . . $=$ . . . , and . . . $=$ . . . . So $\triangle AEF \cong \triangle AFG$ because. . . . Therefore $AB = AC$.

Comparing the analytic and synthetic proofs

To most people the *synthetic proof* is the more convincing, and it is also the form better suited for recording a proof, but it gives no clue to the method by which the proof was discovered. The *analytic proof*, on the other hand, leaves you in no doubt as to where to begin. You know that you must start with the statement to be proved. You know that the next statement must be one of the few methods of proving this conclusion. Consequently, it is much easier to discover a proof by the analytic than by the synthetic method.

Class Exercises

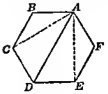
1. Two triangles are congruent, if two sides and the median to one of them in one triangle equal respectively two sides and the corresponding median in the other.

2. If two isosceles triangles have the same base, the line joining their vertices bisects the base.

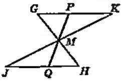
3. Two triangles are congruent, if two sides and the line which joins the middle points of those sides in one triangle equal respectively two sides and the corresponding line in the other.

4. If the opposite sides of a quadrilateral are equal, the middle point of a diagonal bisects any line passing through it and ending in the sides of the quadrilateral.

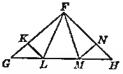
5. If $ABCDEF$ is equilateral and equiangular, the diagonal AD bisects $\angle CDE$.



Ex. 5.



Exs. 6, 7.



Ex. 9.

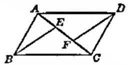
6. If GH and JK bisect each other at M, then PQ is bisected at M.

7. If $GK = JH$, $\angle G = \angle H$, and $\angle J = \angle K$, prove that $PM = MQ$.

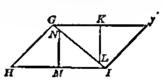
Optional Exercises

8. If polygon $ABCDE$ is equilateral and the diagonals AC and AD are equal, prove that $\angle BCD = \angle CDE$.

9. If equal lengths GK and HN are taken on the legs of isosceles $\triangle FGH$, and at K and N perpendiculars to the legs are drawn meeting the base at L and M, then $FL = FM$.



Ex. 12.



Ex. 13.

10. In the quadrilateral $PQRS$, $PQ = PS$ and $QR = RS$. If T is any point on the diagonal PR, then $TQ = TS$.

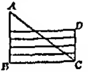
11. If the opposite sides of a quadrilateral are equal, the diagonals bisect each other.

12. If the opposite sides of quadrilateral $ABCD$ are equal, and $AE = FC$, then $BE = FD$.

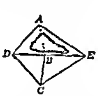
13. If the opposite sides of a quadrilateral are equal, perpendiculars to a pair of opposite sides at their middle points, and ending in a diagonal, are equal.

It is made of several pieces hinged at all lettered points and supported at A and E only. Can it bend at any of these points? Why do you suppose he drew all triangles instead of quadrilaterals or other figures?

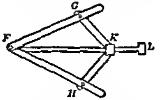
29. $ABCD$ is a gate whose boards are fastened by a single nail at each joint. Why will the strip AC, nailed at A and C, hold the gate rigid?



30. George Shaw told Harold Stern that he could measure the distance from A to B without crossing the pond. Harold did not believe this, so George measured the lengths AD, AE, and DE, and then made DC and CE equal respectively to AD and AE. He then measured BC and found it to be 153 feet. George says that this is the distance from A to B. Is he right? Give proof. Assuming that it requires less work to measure an angle than a long line, can you suggest a better method of finding AB than that which George used?



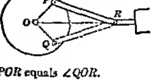
Ex. 30.



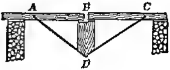
Ex. 31.

31. In the angle bisector, $FG = FH$ and $GK = KH$, and K can slide along the rod FL. To bisect an angle, FG and FH are placed in coincidence with the sides, and a line is drawn along FL. Prove that this line bisects the angle.

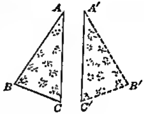
32. The wheel O is turned by the crank-shaft PR. As the wheel turns, point R moves along the line OR. If P and Q are the two positions of the crank pin when R occupies the same position on OR, prove that $\angle POR$ equals $\angle QOR$.



33. An inverted king-post bridge is built as shown in the figure. ADC is a strong wire cable fastened securely at A, D, and C. Explain why the bridge cannot bend at B and fall.



34. Rosemary has made a large triangular flower bed ABC on one side of the walk on her front lawn, and wishes to make another $A'B'C'$ on the other side exactly congruent with it. Show what measurements she must make, and construct a plan of the walk and the two flower beds.



A Self-Measuring Test

There is no reason why you should be in doubt about your ability to do the exercises of this chapter or the work of the next unit. Here are questions which will help you to make an inventory of your equipment. If you can answer all of these questions, you can go forward with confidence.

1. Give three methods for proving triangles congruent.

2. What method of proving lines equal have you learned?

3. Give five ways of showing that angles are equal.

4. Give the definition and state some other fact learned about each of the following: (a) *right angles;* (b) *adjacent angles;* (c) *vertical angles;* (d) *supplementary angles;* (e) an *isosceles triangle.*

5. Why is it that the proposition in which triangles are proved congruent by three sides is not proved in the same way as the other two congruent triangle propositions?

6. Name four sizes of angle and define each.

7. What is meant by *perpendicular lines?*

8. Which part of a proposition is the *hypothesis,* (a) when the sentence has two clauses? (b) when the sentence has but one clause?

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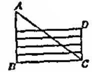

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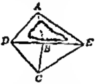
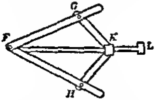


Ex. 30.



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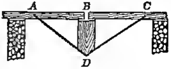

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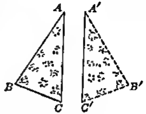

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6. Name four sizes of angle and define each.

7. What is meant by *perpendicular lines?*

8. Which part of a proposition is the *hypothesis,* (*a*) when the sentence has two clauses? (*b*) when the sentence has but one clause?

9. State two facts which you have learned about a straight line.

10. What axioms can be used to prove lines equal? Angles equal?

11. Define and illustrate: (*a*) the *altitude* of a triangle; (*b*) the *median* of a triangle.

12. How many altitudes and how many medians has a triangle?

13. Why is the proposition about the base angles of an isosceles triangle placed between the first two and the third congruent-triangle propositions, instead of having all three together?

14. What is a *polygon*? Is a triangle a polygon?

15. What is meant by an *included angle*? An *included side*?

16. How many parts must be known in order to prove triangles congruent? How many of these can be sides? How many can be angles?

17. In a right triangle, what is the side opposite the right angle called? What are the sides including the right angle called?

18. How many degrees are there in a straight angle? In a right angle?

19. What two instruments are you allowed to use in constructing a figure? What other instruments have you used in drawing a figure?

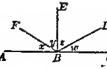
20. Explain the difference between a *synthetic proof* and an *analytic proof*. Which form of proof do you think best suited for discovering the way to prove a theorem?

21. What is an *axiom*? Give three geometric assumptions about lines or angles. Give six axioms that do not mention geometric figures. Give the three postulates.

Method of attack. *To prove a line perpendicular to another line*, show that it makes the two supplementary angles equal. Then use § 12.

In the exercises below, the supplementary angles are adjacent, but do not think that this is always the case. Angles that are both equal and supplementary are right angles wherever they may be.

Exercises



1. If $\angle x = \angle w$ and $\angle y = \angle z$, then $EB \perp AC$.

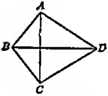
2. If $\angle x = \angle z$, FB bisects $\angle ABE$ and BD bisects $\angle EBC$, then $EB \perp AC$.

3. The bisector of the vertex angle of an isosceles triangle is perpendicular to the base.

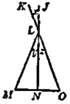
4. The median to the base of an isosceles triangle is perpendicular to the base.

5. Two congruent triangles ABC and $A'B'C'$ are placed beside each other so that AC and $A'C'$ coincide. If BC and $B'C'$ form a straight line, then AC is perpendicular to BC.

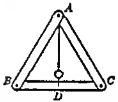
6. If $AB = BC$ and BD bisects $\angle B$, then $BD \perp AC$.



Exs. 6, 7.



Ex. 9.



Ex. 10.

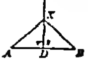
7. If $ABCD$ is folded along BD and it is found that $\triangle ABD$ and CBD coincide, then $AC \perp BD$.

8. The diagonals of a square are perpendicular to each other.

9. If $\angle x = \angle y$ and $LM = LO$, then $LN \perp MO$.

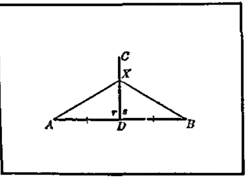
10. A carpenter sometimes determines whether a floor is horizontal by using an instrument like that shown here in which the legs AB and AC are equal, and there is a mark at the middle point D of BC. Show that BC is horizontal ($\perp$ the plumbline) if the plumb bob hangs at D.

Investigation Problem. CD is perpendicular to AB and bisects AB. How does the distance from X to A compare with the distance from X to B? What method do you know of proving two lines equal? Can you prove that XA equals XB without looking at Proposition 5?



Proposition 5

★ 59. *Any point on the perpendicular bisector of a line segment is equally distant from the ends of the segment.*



Given: $AD = DB$, $CD \perp AB$,
and X any point on CD.
To prove: $XA = XB$.

Proof: Statements	Reasons
1. In $\triangle ADX$ and XDB, $AD = DB$.	1. Hyp.
2. $XD = XD$.	2. Iden.
3. $\angle r$ and $\angle s$ are rt. angles.	3. Two lines meeting at rt. angles are $\perp$ to each other.
4. $\angle r = \angle s$.	4. Rt. angles are equal.
5. $\triangle ADX \cong \triangle XDB$.	5. Two ▵ are $\cong$ if two sides and the included angle of one equal . . .
6. $XA = XB$.	6. Corresponding sides of $\cong$ ▵ are equal.

* **60.** Converse. *Any point, equally distant from the ends of a line segment, is on the perpendicular bisector of that segment.*

Given: [Line segment AB and point X]; $AX = XB$.

To prove: That X is on the perpendicular bisector of AB.

Proof: STATEMENTS	REASONS
1. Let D be the middle point of AB.	1. A line segment has a middle point.
2. Draw CD through X.	2. A st. line can be drawn between any two points.
3. In $\triangle ADX$ and XDB, $AD = DB$.	3. D is the middle point of AB.
4. $XD = XD$.	4. Iden.
5. $AX = XB$.	5. Hyp.
6. $\triangle ADX \cong \triangle XDB$.	6. Two $\triangle$ are congruent if three sides of one equal . . .
7. $\angle r = \angle s$.	7. Corr. angles of $\cong$ $\triangle$ are equal.
8. $\angle r$ is a rt. angle.	8. If one st. line meets another st. line so as to make two adjacent angles equal, each of these angles is a rt. angle.
9. $CD \perp AB$.	9. Two lines meeting at rt. angles are $\perp$ to each other.
10. X is on the $\perp$ bisector of AB.	10. $CD \perp AB$ and bisects AB.

61. Corollary. *Two points, each equally distant from the ends of a line segment, determine the perpendicular bisector of that segment.*

Converse. What is the hypothesis in section 59? What is the conclusion? What is the hypothesis in section 60? What is the conclusion? What do you notice about the

hypothesis and conclusion in comparing these two statements? The second statement is called the converse of the first.

The converse of a theorem is another theorem in which the hypothesis and conclusion are interchanged, that is, the hypothesis of one is the conclusion of the other, and vice versa.

State the converse of each of the following statements:

1. If two sides of a triangle are equal, the angles opposite them are equal.
2. If two angles are equal, their supplements are equal.
3. An equilateral triangle is equiangular.
4. Right angles are equal.
5. A squirrel is a small animal.

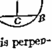
Are all of the above statements true? Are all of the converses true?

We see that the converse of a theorem is not necessarily true because the theorem is true. It may be true or it may be false. Consequently, if we wish to use the converse, we must prove it as a separate proposition.

62. Method of attack. *To prove a line perpendicular to another line,* show that two points on one of them are each equally distant from two points on the other.

CLASS EXERCISES

1. The median to the base of an isosceles triangle is perpendicular to the base.
2. The line joining the vertices of two isosceles triangles on the same base is the perpendicular bisector of the base.



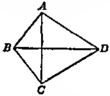
3. A radius to the middle point of a chord of a circle is perpendicular to the chord.

4. The diagonals of a square are perpendicular to each other.

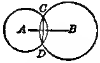
5. The diagonals of a rhombus are perpendicular to each other.

6. In an isosceles triangle the bisector of the vertex angle is perpendicular to the base.

7. If two adjacent sides of a quadrilateral are equal, and the other two sides are equal, the diagonals are perpendicular to each other.



Exs. 7, 8.



Ex. 9

8. If a diagonal of a quadrilateral bisects both angles, it is the perpendicular bisector of the other diagonal.

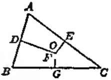
9. *If two circles intersect, the line joining their centers is the perpendicular bisector of their common chord.*

10. If the perpendicular bisector of a side of a triangle passes through the vertex, the triangle is isosceles.

Optional Exercises

11. The perpendicular bisector of the base of an isosceles triangle passes through the vertex.

12. *The perpendicular bisector of a chord passes through the center of the circle.*



13. The point of intersection of the perpendicular bisectors of two sides of a triangle is equally distant from the three vertices.

14. The three perpendicular bisectors of the sides of a triangle meet in a point.

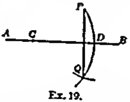
15. Two straight lines from a point in a perpendicular to a line, cutting off on the given line equal lengths from the foot of the perpendicular, are equal.

Honor Work

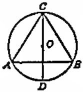
16. A point, not on the perpendicular bisector of a line segment, is unequally distant from the ends of the segment.

17. If three or more isosceles triangles have the same base, their vertices lie on a straight line.

18. If a point moves so that it always remains equally distant from the ends of a segment, it traces the perpendicular bisector of the segment.



Ex. 19.



Ex. 20.

19. To construct a perpendicular from P to AB, take any point C on AB as center, and, with CP as radius, construct an arc PQ, cutting AB at D. Then, with D as center and PD as radius, construct an arc, cutting PQ at Q. Prove that PQ is the required perpendicular.

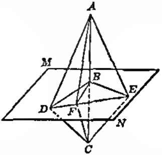
20. If $CA = CB$ and CD passes through the center of circle O, then CD is the perpendicular bisector of AB.

Geometric reasoning applied to life situations (optional)

On all very high mountains there is a line, called the timber line, above which trees will not grow. Are the following conclusions necessarily true? (*a*) Since Mount Blue has no trees on its summit, it extends above the timber line. (*b*) Since Mount Blank has trees on its summit, it does not extend above the timber line. (*c*) Since Mount White extends above the timber line, it has no trees on its summit. (*d*) Since Mount Verde does not extend above the timber line, it has trees on its summit.

SPACE GEOMETRY (*Optional*)

A line perpendicular to a plane. A line AC is perpendicular to a plane MN if it is perpendicular to all the lines of the plane through the point B where it intersects the plane.



1. If $AC \perp BD$ and BE and if $AB = BC$, prove

   (a) That $AD = DC$
   (b) That $AE = EC$
   (c) That $\triangle ADE \cong \triangle CDE$.
   (d) That $AF = FC$.
   (e) That $BF \perp AC$.
   (f) If BF is any line in MN through B, why is $AB \perp MN$?

2. If in the same hypothesis, $BD = BE$, then $AD = AE$.

3. If a line is perpendicular to one line in a plane, must it be perpendicular to the plane? Illustrate, using pencil and cardboard or the top of the desk.

4. In space can more than one line be drawn perpendicular to the same line at the same point of the line?

5. Can more than one line be drawn perpendicular to a plane at the same point of the plane?

6. If a plane is not perpendicular to a line, is there any line in the plane that is perpendicular to the line? Illustrate.

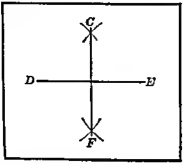
7. To determine that the door jam is perpendicular to the floor, in how many positions must the carpenter place his square?

Investigation Problem. By drawing arcs, find a point equally distant from the ends of a line segment AB. To construct the perpendicular bisector of AB, how many such points are needed? Find them and complete the construction.

Take a point C on line AB. Can you draw arcs cutting off a segment of AB so that C is equally distant from its ends? Will the perpendicular bisector of this segment pass through C? Complete the construction of a perpendicular to AB at C and prove your work. Could you choose a radius too short for this construction?

Proposition 6

63. *The perpendicular bisector of a given line segment can be constructed.*



Given: A line segment DE.

To prove: A perpendicular bisector of DE can be constructed.

Construction: Statements	Reasons
1. With D as center and any convenient radius, construct arcs on both sides of DE.	1. A circle or arc can be constructed with any center and any radius.
2. With E as center and the same radius, intersect these arcs at C and F.	2. Same as No. 1.
3. Draw CF.	3. A st. line can be drawn between any two points.

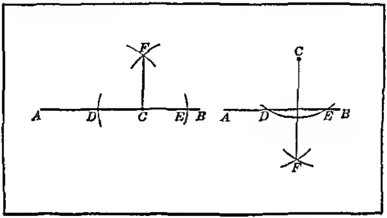
Then CF is the required line.

Proof:

1. C is equally distant from D and E.	1. Const.
2. F is equally distant from D and E.	2. Const.
3. CF is the $\perp$ bisector of DE.	3. Two points, each equally distant from the ends of a line segment, determine the $\perp$ bisector of that segment.

Proposition 7

64. *Through a given point, a perpendicular to a given line can be constructed.*



Given: Line AB and point C.

To prove: A line through $C \perp AB$ can be constructed.

Case 1. *When the point is on the line.*

Case 2. When the point is off the line.

Construction: Statements	Reasons
1. With C as center and any convenient radius, cut AB at D and E.	1. A circle or arc can be constructed with any center and any radius.
2. With D as center and a radius more than one half DE, construct an arc.	2. Same as No. 1.
3. With E as center and the same radius, intersect the arc at F.	3. Same as No. 1.
4. Draw CF.	4. A st. line can be drawn between any two points.

Then CF is the required line.

Proof:

1. C is equally distant from D and E.	1. Const.
2. F is equally distant from D and E.	2. Const.
3. CF is the $\perp$ bisector of DE.	3. Two points, each equally distant from the ends of a line segment, determine the $\perp$ bisector of that segment.

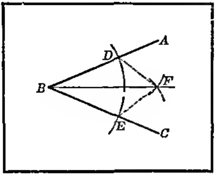
NOTE: All statements, in both construction and proof, for Case 2 are exactly the same as those for Case 1.

Review Exercises

1. Divide a line segment into four equal parts.
2. Construct a line segment twice as long as a given segment.
3. Construct a line segment equal to the sum of two segments.
4. Construct a line segment one and one-half times the length of a given segment.
5. Construct the altitudes of an acute triangle.
6. Construct the altitudes of an obtuse triangle.
7. Construct the perpendicular bisectors of the sides of a triangle.
8. Construct a circle passing through all three vertices of a triangle.
9. Construct the medians of a triangle.
10. Construct a square, given a side.
11. Construct a square, given the perimeter.
12. Construct the complement of a given acute angle.
13. Construct a right triangle whose legs are 2 in. and 3 in.
14. Construct an isosceles right triangle whose leg is 2 in.
15. Construct an isosceles triangle, given the base and altitude.
16. Construct a triangle whose sides are respectively half the length of the sides of a given triangle.
17. Given two unequal lines, a and b, a being longer than b, construct $3a-b$; $a+3b$; $\frac{2a-b}{2}$; $\frac{1}{2}a+\frac{1}{4}b$.

Proposition 8

65. *A given angle can be bisected.*



Given: $\angle ABC$.

To prove: $\angle ABC$ can be bisected.

Construction: Statements	Reasons
1. With B as center, and with any radius, cut AB at D and BC at E.	1. Post. 3.
2. With D and E as centers and with equal radii, draw arcs intersecting at F.	2. Post. 3.
3. Draw BF.	3. Post. 1.

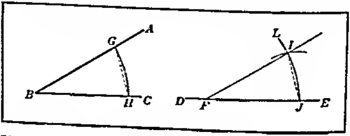
Then BF is the required line.

Proof:

1. Draw DF and EF.	1. Post. 1.
2. In $\triangle DBF$ and EBF, $BF = BF$.	2. Iden.
3. $BD = BE$.	3. Const.
4. $DF = EF$.	4. Const.
5. $\triangle DBF \cong \triangle EBF$.	5. Two $\triangle$ are $\cong$ if the three sides . . . (§ 57).
6. $\angle ABF = \angle CBF$.	6. Corr. parts of $\cong$ $\triangle$ are $=$.
7. BF bisects $\angle ABC$.	7. To bisect is to cut in two equal parts.

PROPOSITION 9

66. *An angle equal to a given angle can be constructed at a given point on a given straight line.*



Given: $\angle B$, line DE, and point F on DE.

To prove: A line can be constructed at F making an $\angle = \angle B$.

Construction:

STATEMENTS	REASONS
1. With B as center, and any radius cut AB at G and BC at H.	1. Post. 3.
2. With F as center and the same radius, construct an arc LJ, cutting DE at J.	2. Post. 3.
3. With J as center and GH as radius, cut the arc LJ at I.	3. Post. 3.
4. Draw FI.	4. Post. 1.

Then $\angle IFE$ is the required angle.

Proof:

1. Draw GH and IJ.	1. Post. 1.
2. In $\triangle BHG$ and FJI, $BG = FI$.	2. Const.
3. $BH = FJ$.	3. Const.
4. $GH = IJ$.	4. Const.
5. $\triangle BHG \cong \triangle FJI$.	5. Two $\triangle$ are $\cong$ if the three sides . . . (§ 57).
6. $\angle B = \angle IFE$.	6. Corr. angles of $\cong$ $\triangle$ are $=$.

Review Exercises

1. Divide a given angle into four equal parts.

2. Construct an angle of 45°, 135°.

3. Construct an angle of 22° 30′, 67° 30′.

4. Construct an angle equal to twice a given acute angle.

5. Construct an angle equal to the sum of two given angles.

6. Construct an angle equal to the difference of two given angles.

7. Construct a triangle having two sides and their included angle equal respectively to two given lines and a given angle.

8. Construct a triangle having two angles and their included side equal respectively to two given angles and a given line.

9. Construct an isosceles triangle, given the vertex angle and a leg.

10. Construct an isosceles triangle, given a base angle and the base.

11. Construct the bisectors of the angles of a given triangle.

12. At a given point on a line, construct a perpendicular to the line by bisecting the straight angle.

13. Construct a $\triangle ABC$, making $AB=3$ in., $\angle A=90°$, and $\angle B=45°$.

14. Construct a $\triangle ABC$, making $AB=4$ in., $BC=3$ in., and $\angle B=22° 30'$.

15. Construct a triangle congruent to a given triangle.

16. Construct a triangle whose base angles equal those of a given triangle, but whose base is one and one-half times as long.

True-False Test (*10 min.*)

Copy the numbers of the statements. If the statement is true, write T after its number, if false, write F.

1. In proving the base angles of an isosceles triangle equal, we construct the altitude to the base.

2. An analytic proof is one in which we begin with the statement to be proved.

3. If two triangles are congruent, their corresponding angles are equal.

4. Supplements of equal angles are equal.

5. A triangle is acute if it has an acute angle.

6. A perpendicular is a line running up and down.

7. An equilateral triangle is equiangular.

8. Two triangles are congruent if two sides and an angle of one equal two sides and an angle of the other.

9. To bisect means to cut in two equal parts.

10. Two triangles are congruent if two angles and a side of one equal two angles and a side of the other.

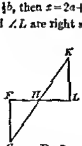
Reasoning Test (*10 min.*)

Think how you would prove each of the following statements. Then copy the numbers of these statements and after each number write only the final reason used in the proof. Do not write out the proof in full.

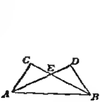
1. Two right triangles are congruent if the legs of one equal the legs of the other.

2. If $\frac{1}{2}x = a + \frac{1}{2}b$, then $x = 2a + b$.

3. If $\angle F$ and $\angle L$ are right angles and $FH = HL$, then $\triangle FGH \cong \triangle KLH$.



Ex. 3.



Ex. 5.

4. If $x + 7 = 16$, then $x = 9$.

5. If $\angle CAB = \angle DBA$ and AD and CB are angle bisectors, then $\triangle ACB \cong \triangle BDA$.

Multiple-Choice Test (*10 min.*)

From the four answers given, select the one which will make the statement true.

1. If a diagonal is drawn in a quadrilateral whose opposite sides are equal, the triangles formed can be proved congruent by, (*a*) s.a.s.; (*b*) a.s a.; (*c*) s.s.s.; (*d*) corr. $\angle$s.

2. In proving that the construction for bisecting an angle is correct, the triangles are proved congruent by the method, (*a*) rt. $\angle$s are equal; (*b*) s.a.s.; (c) a.s.a.; (*d*) s.s.s.

3. If the bisector of an angle of a triangle is perpendicular to the opposite side, the two triangles formed can be proved congruent by, (*a*) s.a.s.; (*b*) a.s.a.; (c) vert. $\angle$s are equal; (*d*) s.s.s.

4. If the three angles of one triangle equal the three angles of another triangle, the triangles are, (*a*) congruent; (*b*) equilateral; (*c*) not necessarily congruent; (*d*) isosceles.

5. *To construct the perpendicular to a line segment through a given point, we must first find, (a) 0; (b) 1; (c) 2; (d) 3, of its points.*

Construction Test (*10 min.*)

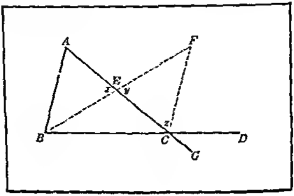
1. Construct an equilateral triangle, given a side n.
2. Construct a median of a given triangle.
3. Construct a perpendicular to a line at a point on the line.
4. Divide a line segment into four equal parts.
5. Construct an angle one and one-half times a given angle.

67. *An exterior angle of a triangle* is an angle formed by one side of the triangle and the prolongation of another side through their common point, as $\angle ACD$.

One angle of the triangle is adjacent to the exterior angle. The other two angles are called remote interior angles. $\angle ACB$ is adjacent to $\angle ACD$. $\angle A$ and $\angle B$ are the remote interior angles.

PROPOSITION 10

68. *An exterior angle of a triangle is larger than either remote interior angle.*



Given: $\triangle ABC$ with exterior $\angle ACD$.

To prove: $\angle ACD > \angle A$ or $\angle B$.

Proof: STATEMENTS	REASONS
1. Bisect AC at E.	1. § 63.
2. Draw BE, and extend it making $EF = BE$.	2. Post. 1 and 2.
3. Draw CF.	3. Post. 1.
4. In $\triangle ABE$ and CEF, $AE = EC$.	4. Const.
5. $BE = EF$.	5. Const.
6. $\angle x = \angle y$.	6. § 37.
7. $\triangle ABE \cong \triangle CEF$.	7. § 50.
8. $\angle A = \angle z$.	8. § 54.
9. $\angle ACD > \angle z$.	9. Ax. 8.
10. $\angle ACD > \angle A$.	10. Subst.

In like manner, it can be shown that $\angle BCG > \angle B$.

11. $\angle BCG = \angle ACD$.	11. § 37.
12. $\angle ACD > \angle B$.	12. Subst.

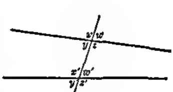
NOTE TO THE STUDENT: From this point on, you will often find a number in place of the reason for a statement. This number refers to the paragraph in which the reason can be found. Never turn back to that paragraph until you have tried to think out the reason for yourself. In fact, a good student will never read either statement or reason until he has first tried to prove the proposition himself. Are you good enough student for that?

PARALLEL LINES

69. A transversal is a line that cuts two or more other lines.

70. If two straight lines are cut by a transversal, the angles are named as follows:

$\angle s\ w', x', y, z$, are interior $\angle s$.
$\angle s\ w, x, y', z'$, are exterior $\angle s$.



A pair of angles are alternate, when they are on opposite sides of the transversal, and one at each vertex, as $\angle s\ y$ and w', z and x', w and y', and x and z'.

The pairs y and w', and z and x' are alternate interior angles.

The pairs w and y', and x and z' are alternate exterior angles.

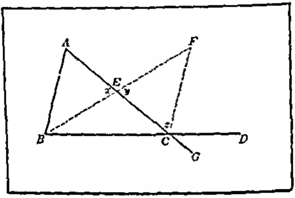
The pairs w and w', x and x', y and y', and z and z', are corresponding angles.

When alternate interior angles, or any of the above pairs of angles are mentioned, it is understood that there are two straight lines cut by a transversal.

71. Parallel lines are straight lines, in the same plane, that cannot meet, however far extended in either direction.

Proposition 10

68. *An exterior angle of a triangle is larger than either remote interior angle.*



Given: $\triangle ABC$ with exterior $\angle ACD$.
To prove: $\angle ACD > \angle A$ or $\angle B$.

Proof: Statements	Reasons
1. Bisect AC at E.	1. § 63.
2. Draw BE, and extend it making $EF = BE$.	2. Post. 1 and 2.
3. Draw CF.	3. Post. 1.
4. In $\triangle ABE$ and CEF, $AE = EC$.	4. Const.
5. $BE = EF$.	5. Const.
6. $\angle x = \angle y$.	6. § 37.
7. $\triangle ABE \cong \triangle CEF$.	7. § 50.
8. $\angle A = \angle z$.	8. § 54.
9. $\angle ACD > \angle z$.	9. Ax. 8.
10. $\angle ACD > \angle A$.	10. Subst.

In like manner, it can be shown that $\angle BCG > \angle B$.

11. $\angle BCG = \angle ACD$.	11. § 37.
12. $\angle ACD > \angle B$.	12. Subst.

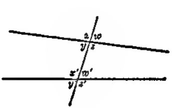
NOTE TO THE STUDENT: From this point on, you will often find a number in place of the reason for a statement. This number refers to the paragraph in which the reason can be found. Never turn back to that paragraph until you have tried to think out the reason for yourself. In fact, a good student will never read either statement or reason until he has first tried to prove the proposition himself. Are you good enough student for that?

PARALLEL LINES

69. A **transversal** is a line that cuts two or more other lines.

70. If two straight lines are cut by a transversal, the angles are named as follows:

$\angle$s w', x', y, z, are **interior** $\angle$s.
$\angle$s w, x, y', z', are **exterior** $\angle$s.



A pair of angles are **alternate**, when they are on opposite sides of the transversal, and one at each vertex, as $\angle$s y and w', z and x', w and y', and x and z'.

The pairs y and w', and z and x' are **alternate interior angles.**

The pairs w and y', and x and z' are **alternate exterior angles.**

The pairs w and w', x and x', y and y', and z and z', are **corresponding angles.**

When alternate interior angles, or any of the above pairs of angles are mentioned, it is understood that there are two straight lines cut by a transversal.

71. **Parallel lines** are straight lines, in the same plane, that cannot meet, however far extended in either direction.

DO PARALLEL LINES MEET?

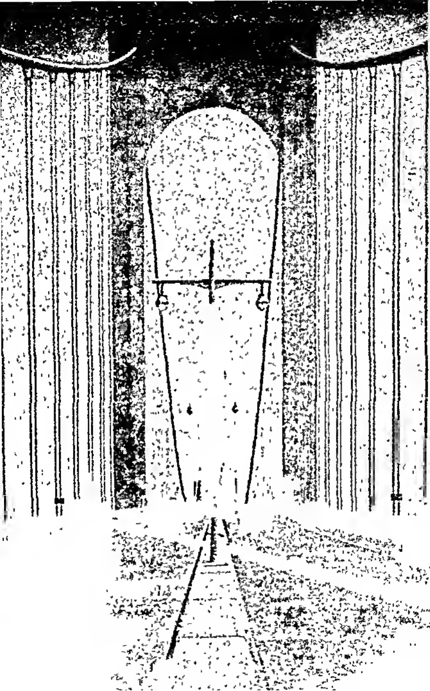
Notice the parallel lines in the picture. Do they look parallel? First look at the vertical cables that support the bridge. Do they look parallel? Now look at the edges of the center strip and the curbs of the bridge. Are they parallel? Do they look parallel? Why do some parallel lines look parallel whereas others do not?

In order to make this picture look natural, the artist must take account of this peculiar fact that some parallel lines appear to meet. He says that they meet at infinity and calls the line along which they appear to meet the line at infinity. Ancient painters did not understand this. If you visit an art gallery, you will find that very early paintings do not look real because they have no line at infinity.

See if you can find other parallel lines both in your classroom and on your way to and from school. Do they look parallel from your position? Can you discover in what direction parallel lines must run so that they will look parallel? So they will appear to meet?

Unlike the triangle, the cable does not hold its shape under pressure, but it has the greater strength for resisting a straight pull. In the type of bridge shown in this picture the weight is a direct pull downward. So the cable gives the greatest strength. And because this pull is directly downward, the small cables are all vertical parallel lines.

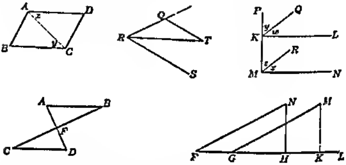
Photo by George A. Douglas from Philip D. Gendreau



Axiom 11. Through a point not more than one line can be drawn parallel to a given line.

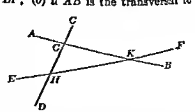
Exercises

1. Name all pairs of alternate interior angles and corresponding angles in the following figures:



2. What pairs of angles are made by the letter H? N? F?

3. In the figure, name a pair of alternate interior angles, (*a*) if CD is the transversal to AB and EF; (*b*) if AB is the transversal to CD and EF; (*c*) if EF is the transversal to AB and CD.



Investigation Problem. If, in the figure in § 70, $\angle y$ equaled $\angle w'$, what would you suspect about the lines? Suppose they crossed at K as AB and EF do in the figure for Ex. 3 above, what kind of figure would GHK be? Is $\angle AGH$ an angle of this triangle? What do we call $\angle AGH$ with reference to the triangle? How does it compare in size with $\angle GHK$? Why? If the lines cross at some point K to the right of CD, must $\angle AGH$ necessarily be larger than $\angle GHK$? If, when you knew that $\angle AGH$ were equal to $\angle GHK$, what could you say about the lines crossing? And, if they cannot cross, what do you say about them?

PROPOSITION 11

72. *Two lines are parallel if their alternate interior angles are equal.*

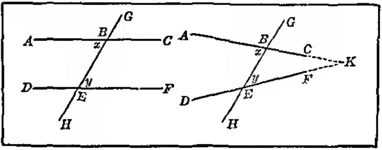


FIG. 1. FIG. 2.

Given: [AC and DF cut by GH], $\angle x = \angle y$.
To prove: $AC \| DF$.

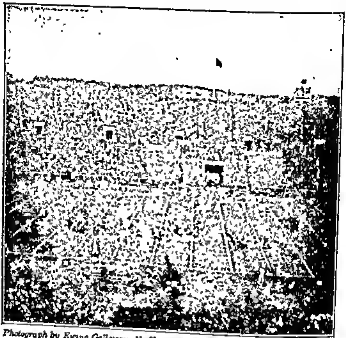
Proof: STATEMENTS	REASONS
1. Either $AC \| DF$ as in Fig. 1, or they will meet at some point K if extended as in Fig. 2.	1. § 71.
2. If they met at K, we would have a $\triangle BEK$ and $\angle x$ would be $> \angle y$.	2. § 68.
3. This is impossible, that is, AC and DF cannot meet.	3. $\angle x = \angle y$ by hyp.
4. $AC \| DF$.	4. § 71.

73. Corollary. *Two lines are parallel if their interior angles on the same side of a tranversal are supplementary.*

Given that $\angle x$ is the supplement of $\angle DEB$.

AC and DF will be parallel if what angles are equal? What is the relation of $\angle x$ to $\angle DEB$? Of $\angle y$ to $\angle DEB$? Of $\angle x$ to $\angle y$?

Indirect proof. The method of proof used in Proposition 11 is called the *indirect proof*. It consists in examining all the



Photograph by Ewing Galloway, N. Y.

PARALLEL LINES.

possibilities, and showing that no other, but the one we wish to prove, could be true. If there are only two possible suppositions, as in Proposition 11, that is, if one or the other of two things must be true, then to prove that one of them is true, it is only necessary to show that the other leads to a contradiction. When there are three suppositions, it is necessary to show that two of them are impossible, thus leaving only one supposition which must therefore be the true one. In another form of indirect proof, seen in Proposition 14, we construct a line which does what we wish to prove a given line does, and then show that the two lines must coincide.

Class Exercises

Use the indirect method. Assume that the conclusion is not true and then show that this leads to an impossible result.

1. If two angles of a triangle are unequal, the sides opposite them are unequal.

2. If any angle of one triangle is not equal to some angle of another triangle, the triangles are not congruent.

3. A point unequally distant from the ends of a line segment is not on the perpendicular bisector of that segment.

4. State and prove the converse of exercise 3.

5. A line segment has only one middle point (assume that there are two middle points).

6. Two lines parallel to the same line are parallel.

7. If a triangle is not isosceles, the bisector of an angle is not perpendicular to the opposite side.

8. If a straight line cuts one of two parallel lines, it cuts the other.

9. Two sides of one triangle equal two sides of another triangle. If their included angles are not equal, their third sides are unequal.

10. Not more than one line can be drawn from a point perpendicular to a given line. (If there are two, compare an exterior angle of the triangle formed with a remote interior angle.)

Optional Exercises

Geometric reasoning applied to life situations.

The indirect proof is one of the best ways of convincing a person that he is wrong.

Illustration: A customer returned the radio you sold him saying that the loud speaker rattles. You try a new detector tube and it works quietly. Convince him by the indirect method that it was not the fault of the loud speaker.

Proof: "If it were the fault of the loud speaker, changing the detector tube would not correct it, would it?" "No." "But you see it does correct it, don't you?" "Yes." "Then it is not the fault of the loud speaker."

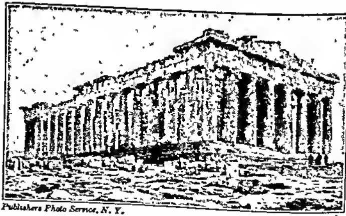
11. When the steam was on last night, water leaked from your radiator on the floor. The janitor says that the radiator is all right. Convince him by the indirect proof that he is wrong.

12. A very heavy package was stolen from John's room. From the time it was seen there until it was missed, it is known that only Peter and the crippled Henry had been in there. You are the prosecuting attorney. By an indirect proof, convince the jury that Peter is guilty.

13. The clock you sold yesterday has stopped. The customer returns it saying that her maid wound it last night but it will not run. You find that the spring is entirely unwound so you wind it and it runs. The spring would be unwound if the clock had run down or if the spring had broken. Convince the lady that her maid forgot to wind the clock.

14. You are the doctor. The child has a fever, rash, sore throat and his tongue is red. He was vaccinated last year.

In measles there is fever, rash, tongue coated white, no sore throat. In scarlet fever there is fever, rash, tongue red and sore throat. In smallpox there is fever and rash but the disease is prevented by vaccination. In other diseases there is no rash. By the indirect method, diagnose the case.



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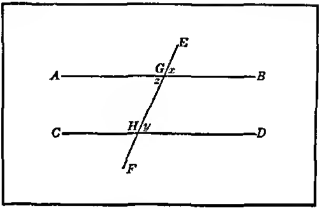
THE PARTHENON AT ATHENS

Notice the use made of parallel lines in this classic temple.

PROPOSITION 12

74. *Two lines are parallel if their corresponding angles are equal.*



Given: [AB and CD cut by EF]; $\angle x = \angle y$.
To prove: $AB \| CD$.

Proof: STATEMENTS	REASONS
1. $\angle x = \angle y$.	1. Hyp.
2. $\angle x = \angle z$.	2. § 37.
3. $\angle y = \angle z$.	3. Subst.
4. $AB \| CD$.	4. § 72.

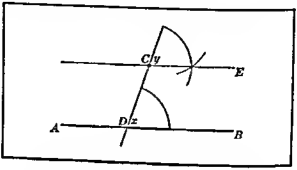
75. Corollary 1. *Lines perpendicular to the same line are parallel.*

* **76. Corollary 2.** *Only one perpendicular can be drawn from a point to a line.* (Use § 75.)

Investigation Problem. If line AB were omitted from the above figure, could you construct an angle x so as to obtain a line through G parallel to CD? Try to complete and prove this construction before looking at Proposition 13.

Proposition 13

77. *Through a given outside point a line parallel to a given line can be constructed.*



Given: Line AB and point C not on AB.

To prove: A line through C parallel to AB can be constructed.

Construction: Statements	Reasons
1. Through C draw CD to any point D on AB.	1. Post. 1.
2. Construct CE, making $\angle y = \angle x$.	2. § 66.

Then CE is the required line.

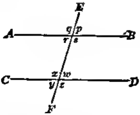
Proof:

1. $\angle y = \angle x$.	1. Const.
2. $CE \parallel AB$.	2. § 74.

78. Method of attack. *To prove lines parallel,* show that a pair of alternate interior angles are equal, or that a pair of corresponding angles are equal.

What other methods that you have studied might be used to prove lines parallel?

Class Exercises



1. Prove $AB \parallel CD$ if:

(a) $\angle p = 65°$ and $\angle w = 65°$.
(b) $\angle p = 65°$ and $\angle x = 115°$.
(c) $\angle p = 70°$ and $\angle y = 70°$.
(d) $\angle p = 75°$ and $\angle z = 105°$.
(e) $\angle r = m°$ and $\angle x = 180° - m°$.

2. Two lines are parallel if the alternate exterior angles are equal.

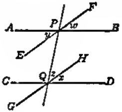
3. By means of alternate interior angles, construct a line through a point parallel to a given line.

4. If $\angle w = \angle z$ and $\angle x = \angle y$, then $AB \parallel FD$.

5. If $AB \perp BF$ and $DF \perp BF$, BH bisects $\angle FBA$, and FC bisects $\angle BFD$, then BH and CF are parallel.

Exs. 4, 5.

Exs. 7, 8.



Ex. 9.

6. If GH and KL bisect each other, prove that $GK \parallel LH$.

7. If the opposite sides of quadrilateral $ABCD$ are equal, they are parallel.

8. If AD and BC are equal and $\angle x = \angle y$, then AB is parallel to DC.

9. Prove $EF \parallel GH$ if:

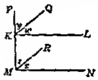
(a) $\angle APQ = 80°$, $\angle w = 50°$, and $\angle z = 30°$.
(b) $\angle APQ = 80°$, $\angle y = 35°$, $\angle x = 35°$, $\angle PQD = 80°$.
(c) $\angle APQ = 85°$, $\angle w = 50°$, $\angle x = 50°$, $\angle PQD = 85°$.
(d) $\angle APQ = 85°$, $\angle y = 40°$, $\angle CQP = 95°$, $\angle x = 40°$.
(e) $\angle APQ = m°$, $\angle y = n°$, and $\angle z = m° - n°$.

10. If KL and MN are perpendicular to PM, KQ bisects $\angle PKL$ and MR bisects $\angle KMN$, then $KQ \parallel MR$.

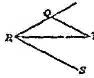
11. If, in the same figure, $\angle PKL = \angle KMN$ and $\angle w = \angle x$, then $KQ \parallel MR$.

12. If RT bisects $\angle QRS$ and $QT = QR$, then $QT \parallel RS$.

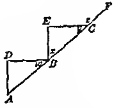
13. If $AD \perp DB$, $BE \perp EC$, $AD = BE$ and $DB = EC$, prove that $AD \parallel BE$ and $DB \parallel EC$.



Exs. 10, 11.



Ex. 12.

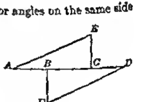


Exs. 13, 14.

14. If $AD \perp DB$, $BE \perp EC$, $DB = EC$, and $\angle DBC = \angle z$, then $AD \parallel BE$.

Optional Exercises

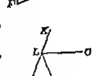
15. Two lines are parallel, if the exterior angles on the same side of the transversal are supplementary.



16. If $AB = CD$, $EC = BF$, and EC and BF are perpendicular to AD, then $AE \parallel DF$.

17. If $AB = CD$, $\angle A = \angle D$ and $AE = DF$, then $EC \parallel BF$.

18. If $\angle A = \angle D$, $AE = DF$, and $\angle E = \angle F$, then $EC \parallel BF$.



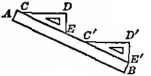
19. If $AB = CD$, $AE = DF$, and $EC = BF$, then $AE \parallel DF$.

20. If $\angle KLN$ is twice $\angle M$, prove that LO, the bisector of $\angle KLN$, is parallel to MP.

21. If $LM = LN$ and $\angle KLO = \angle LNM$, prove $LO \parallel MP$.

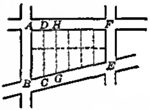
Applied Problems

22. A celluloid triangle CDE is placed against a ruler AB, and a line CD is drawn. Then, holding the ruler fixed, the triangle is slid to the position $C'D'E'$ and a line $C'D'$ is drawn. Prove that $C'D'$ is parallel to CD.

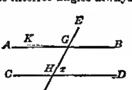


23. The T-square is much used by draughtsmen. Prove that lines, drawn with it as shown in the figure, are parallel.

24. Street BE meets street AB at an angle of 72°. A real-estate agent wishes to cut the block $ABEF$ into lots by constructing lines parallel to street AB. How many degrees must he make angles BCD, BGH, etc.?



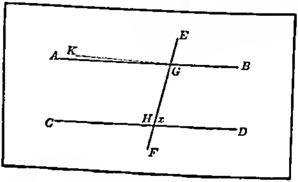
Investigation Problem. State the converse of Proposition 11. Does it appear to be true? Draw two parallel lines. Now draw several transversals in different directions. Are the alternate interior angles always equal? If you draw KG so that $\angle KGH$ equals $\angle x$, what is the relation of KG and CD? Why? What is the relation of AB and CD? Why? Can KG and AB then be separate intersecting lines? Why? If not, what do you know about $\angle AGH$ and $\angle KGH$? About $\angle AGH$ and $\angle x$? Now try to give the complete proof of your conclusion without reading Proposition 14.



The type of proof suggested in this Investigation Problem was mentioned on page 71. In it, we make the line KG do what we wish to prove AB does, and then show that KG and AB are the same line. This method is particularly convenient whenever we wish to prove that a line passes through a certain point, as for example in the proof of §60.

Proposition 14

* 79. *Alternate interior angles of parallel lines are equal.*



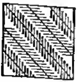
Given: [AB and CD cut by EF]; $AB \parallel CD$.
To prove: $\angle AGH = \angle x$.

Proof:

Statements	Reasons
1. Construct KG, making $\angle KGH = \angle x$.	1. § 66.
2. $KG \parallel CD$.	2. § 72.
3. $AB \parallel CD$.	3. Hyp.
4. KG coincides with AB.	4. Ax. 11.
5. $\angle AGH = \angle KGH$.	5. § 21.
6. $\angle AGH = \angle x$.	6. Subst.

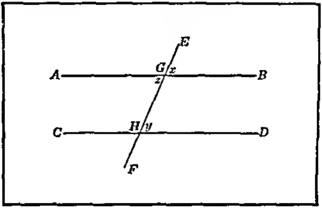
Drawing Exercise

Are the long lines in this square parallel? Make measurements to determine this. Make a larger copy of this figure on paper and try it on your friends.



Proposition 15

80. *Corresponding angles of parallel lines are equal.*



Given: [AB and CD cut by EF]; $AB \parallel CD$.
To prove: $\angle x = \angle y$.

Proof: Statements	Reasons
1. $AB \parallel CD$.	1. Hyp.
2. $\angle y = \angle z$.	2. § 79.
3. $\angle x = \angle z$.	3. § 37.
4. $\angle x = \angle y$.	4. Subst.

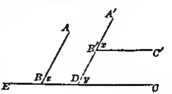
81. Corollary 1. *A line perpendicular to one of two parallel lines is perpendicular to the other.*

82. Corollary 2. *Lines parallel to the same line are parallel.*

83. Corollary 3. *If two lines are parallel, interior angles on the same side of the transversal are supplementary.*

84. Corollary 4. *Angles, having their sides respectively parallel, are either equal or supplementary.*

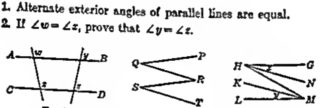
85. Corollary 5. *Angles having their sides respectively perpendicular, are either equal or supplementary.*



Method of attack. *Angles may be proved equal* by showing that they are alternate interior angles or corresponding angles of parallel lines.

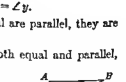
Class Exercises

1. Alternate exterior angles of parallel lines are equal.
2. If $\angle w = \angle x$, prove that $\angle y = \angle z$.



Ex. 2. Ex. 3. Exs. 4, 5.

3. If $PQ \parallel RS$ and $QR \parallel ST$, then $\angle Q = \angle S$.
4. If $GH \parallel LM$ and $\angle x = \angle y$, then $HN \parallel KM$.
5. If $GH \parallel LM$ and $HN \parallel KM$, then $\angle x = \angle y$.
6. If the opposite sides of a quadrilateral are parallel, they are equal.
7. If two sides of a quadrilateral are both equal and parallel, the other two sides are equal and parallel.
8. If two lines are parallel, the exterior angles on the same side of the transversal are supplementary.

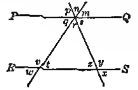


9. If two lines are parallel, the bisectors of a pair of corresponding angles are parallel.
10. If $AB = CD$ and $AB \parallel CD$, then AD and BC bisect each other

11. If $AB \parallel CD$ and $AF = FD$, then $BF = FC$. (Fig. for Ex. 10.)

12. If $PQ \parallel RS$ and $\angle m = \angle p$, prove that:

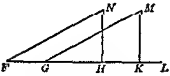
(a) $\angle p = \angle x$
(b) $\angle m = \angle z$
(c) $\angle y = \angle p + \angle n$
(d) $\angle t = \angle z$
(e) $\angle v$ is supp. to $\angle s$
(f) $\angle w = \angle z$
(g) $\angle m$ is supp. to $\angle v$
(h) $\angle m + \angle n + \angle p = \angle r + \angle t + \angle z$



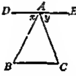
13. If $FG = HK$, $NH \parallel MK$, and $NH = MK$, then $FN \parallel GM$.

14. If $FG = HK$, $NH \parallel MK$, and $FN \parallel GM$, then $FN = GM$.

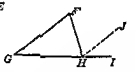
15. If $FG = HK$, $FN = GM$, $FN \parallel GM$, and $NH \perp FL$, then $MK \perp FL$.



Exs. 13, 14, 15.



Exs. 16, 17.



Exs. 18, 19.

16. If $\triangle ABC$ is isosceles and DAE is parallel to the base BC, prove that $\angle x = \angle y$.

17. Prove that the sum of the angles of $\triangle ABC$ equals a straight angle.

18. Prove that the exterior angle FHI equals the sum of $\angle$s F and G.

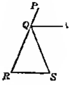
19. Using the figure of Ex. 18, prove that the sum of the angles of a triangle equals a straight angle.

Optional Exercises

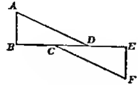
20. In the quadrilateral $KLMN$, $KN \parallel LM$ and $KL \parallel NM$. If $\angle L = 54°$, find the number of degrees in each of the other angles.

21. In isosceles $\triangle QRS$, QV is parallel to the base RS. Prove that QV bisects $\angle PQS$.

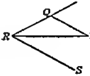
22. If the bisector QV of $\angle PQS$ is parallel to the base RS, then $\angle R = \angle S$.



Exs. 21, 22



Exs. 23, 24, 25, 26



Ex. 27.

23. If $AB \perp BE$, $EF \perp BE$, $BC = DE$, and $AD \parallel CF$, then $\angle A = \angle F$.

24. If $AD = CF$, $AD \parallel CF$, and $BC = DE$, then $AB \parallel EF$.

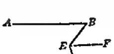
25. If $\angle A = \angle F$, $AD = CF$, and $AD \parallel CF$, then $BC = DE$.

26. If $AB \parallel EF$, $AD \parallel CF$, and $BC = DE$, then $AB = EF$.

27. If TQ equals RQ and is parallel to RS, show that RT bisects $\angle QRS$.

28. Lines perpendicular to parallel lines are parallel.

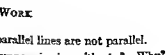
29. If $AB \parallel CD$, prove that $\angle BED = \angle B + \angle D$.



Honor Work

30. Lines perpendicular to non-parallel lines are not parallel.

31. In the figure, how does $\angle y$ compare in size with $\angle x$? Why? If the line AB turns around point C, so that $\angle y$ grows smaller, how does the point of intersection G move? When $\angle y$ becomes almost as small as $\angle x$, where is G? When $\angle y$ becomes equal to $\angle x$? When $\angle y$ becomes smaller than $\angle x$?,

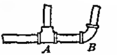


32. As point G moves, in the above example, what change takes place in the size of $\angle CGE$? (Fig. for Ex. 31.)

33. From a given point not on a given line, construct a line making a given angle with the given line.

Applied Problems

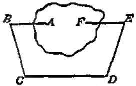
34. A plumber wishes to run two parallel pipes at right angles to the main pipe. He uses two joints, but finds that, while joint A makes a 90° turn, joint B turns only 85°. Prove that, if he uses both joints, the pipes will not be parallel.



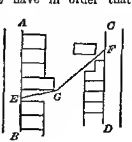
35. In building the gable end of a barn, the studs are all run vertically. Prove that they must all be cut at the same angle to fit against the rafter.

36. If the angle made by the rafters at the ridge is 80°, at what angle are the upright studs cut?

37. In tunneling under a mountain from B to E, the direction BA has been determined so that, if extended, it will pass through E. But in order to work from the other side of the mountain also, the contractor must determine the direction from E. He measures $\angle B$ 77°, $\angle C$ 103° and $\angle D$ 118°. How many degrees must $\angle E$ have in order that EF will meet BA in a straight line?



38. A surveyor wishes to lay out street CD parallel to street AB by constructing equal alternate interior angles at E and F, but as buildings are in the way, he makes $\angle AEG$ 86°, $\angle EGF$ 165°, and extends GF to F. What size must he make $\angle GFD$ in order that CD will be parallel to AB?



SPACE GEOMETRY (*Optional*)

Parallel planes. Two planes are parallel if they can never meet however far extended.

Perpendicular planes. Two planes are perpendicular if they make equal adjacent angles with each other.

The angles made by planes are called *dihedral* angles. They are equal, supplementary, vertical, right, etc., in much the same way as the angles we have already studied.

EXERCISES

In space:

1. Are two lines parallel if, (*a*) Their alternate interior angles are equal; (*b*) their corresponding angles are equal?

2. Are lines perpendicular to the same line parallel?

3. Can more than one line be drawn perpendicular to a given line through a given point, (*a*) if the point is on the line; (*b*) if the point is not on the line?

4. Are alternate interior angles of parallel lines equal?

5. Is a line perpendicular to one of two parallel lines perpendicular to the other?

6. Are angles having their sides respectively parallel equal or supplementary?

7. Are two planes parallel if they are:
   (*a*) Parallel to the same plane?
   (*b*) Perpendicular to the same plane?
   (*c*) Parallel to the same line?
   (*d*) Perpendicular to the same line?

8. Are two lines parallel if they are:
   (*a*) Parallel to the same line?
   (*b*) Perpendicular to the same line?
   (*c*) Parallel to the same plane?
   (*d*) Perpendicular to the same plane?

9. Are a plane and a line not in the plane parallel if they are:

(a) Parallel to the same line?
(b) Parallel to the same plane?
(c) Perpendicular to the same plane?
(d) Perpendicular to the same line?

10. Can two planes perpendicular to the same plane be, (a) parallel; (b) perpendicular?

11. A line is parallel to a plane.

(a) Is a plane perpendicular to the line perpendicular to the plane?
(b) Is a line perpendicular to the line perpendicular to the plane?

12. If one of two parallel lines is perpendicular to a plane, must the other be perpendicular to the plane?

13. If two planes are parallel:

(a) Is a line parallel to one parallel to the other?
(b) Is a plane parallel to one parallel to the other?

14. If a line is parallel to a plane:

(a) Is it parallel to another line that is parallel to the plane?
(b) Is a plane that is parallel to the line also parallel to the plane?

15. If two lines are parallel, is:

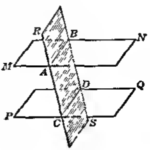
(a) A line that is parallel to one parallel to the other?
(b) A plane that is parallel to one parallel to the other?

16. Are there lines that are neither parallel nor intersecting? Illustrate.

17. The intersections of two parallel planes with a third plane are parallel.

Can AB and CD meet? Explain.

Are AB and CD in the same plane? If so, what plane?



A Self-Measuring Test

1. Give eight methods of proving angles equal.

2. Define *parallel lines.*

3. Give two methods of proving lines parallel.

4. In order to construct a straight line, how many points must be known?

5. How many points of the line must be found in constructing:
   (a) The perpendicular bisector of a line?
   (b) A perpendicular at a point on a line?
   (c) The bisector of an angle?

6. What is the method of finding a point equally distant from two given points?

7. Give the three methods of proving triangles congruent.

8. What are the two principal objects of proving triangles congruent?

9. State the parallel-line axiom. Does this axiom state that one parallel line can be drawn?

10. How do you know that there can be a line through a point parallel to a given line?

11. What is the principal method of proving one line perpendicular to another?

12. What is meant by an *indirect proof*?

13. If parallel lines were defined as two lines whose alternate interior angles are equal, could you prove that parallel lines would never meet?

14. What are *alternate interior angles? Corresponding angles*?

15. Can you give an indirect proof that lines are parallel if the corresponding angles are equal? Why is that method not generally used?

16. Compare the size of the exterior angle of a triangle with that of each of the interior angles.

17. Why is the proposition about the exterior angle of a triangle placed before the parallel-line propositions?

18. Why is it that the construction of a line parallel to another is not placed with the other construction propositions?

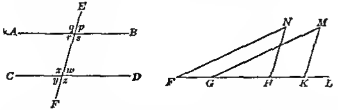
MATCHING TEST (*10 min.*)

Write the numbers 1 to 10 in a column. After each write the letter of the phrase that is the correct definition of the word following the number.

1. Analysis	*a.* A proof in which we show that other possibilities are wrong.
2. Alt. int. ∠s	*b.* A proof in which we begin with the statement to be proved.
3. Transversal	*c.* Angles between two parallel lines and on opposite sides of a transversal.
4. Indirect proof	*d.* An angle formed by extending one side of a triangle.
5. Postulate	*e.* Lines in the same plane that cannot meet.
6. Exterior ∠	*f.* Lines that meet at right angles.
7. Median	*g.* A line cutting two or more other lines.
8. Bisector	*h.* A line from a vertex of a triangle to the middle point of the opposite side.
9. Perpendicular lines	*i.* A construction admitted as possible without proof.
10. Parallel lines	*j.* A figure that cuts another figure in two equal parts.

REASONING TEST (*10 min.*)

If the conclusion is true, copy those statements from the hypothesis used in proving it, if false, write F.



1. If $\angle r=70°$, $AB=CD$, $\angle s=110°$ and EF is a st. line, then $AB \parallel CD$.

2. If $\angle p=\angle r$, $\angle q=$supp. of $\angle y$ and $EF=CD$, then $AB \parallel CD$.

A Self-Measuring Test

1. Give eight methods of proving angles equal.

2. Define *parallel lines.*

3. Give two methods of proving lines parallel.

4. In order to construct a straight line, how many points must be known?

5. How many points of the line must be found in constructing:

(a) The perpendicular bisector of a line?

(b) A perpendicular at a point on a line?

(c) The bisector of an angle?

6. What is the method of finding a point equally distant from two given points?

7. Give the three methods of proving triangles congruent.

8. What are the two principal objects of proving triangles congruent?

9. State the parallel-line axiom. Does this axiom state that one parallel line can be drawn?

10. How do you know that there can be a line through a point parallel to a given line?

11. What is the principal method of proving one line perpendicular to another?

12. What is meant by an *indirect proof?*

13. If parallel lines were defined as two lines whose alternate interior angles are equal, could you prove that parallel lines would never meet?

14. What are *alternate interior angles? Corresponding angles?*

15. Can you give an indirect proof that lines are parallel if the corresponding angles are equal? Why is that method not generally used?

16. Compare the size of the exterior angle of a triangle with that of each of the interior angles.

17. Why is the proposition about the exterior angle of a triangle placed before the parallel-line propositions?

18. Why is it that the construction of a line parallel to another is not placed with the other construction propositions?

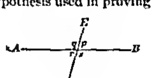
MATCHING TEST (*10 min.*)

Write the numbers **1** *to* **10** *in a column. After each write the letter of the phrase that is the correct definition of the word following the number.*

1. Analysis	*a.* A proof in which we show that other possibilities are wrong.
2. Alt. int. ∠s	*b.* A proof in which we begin with the statement to be proved.
3. Transversal	*c.* Angles between two parallel lines and on opposite sides of a transversal.
4. Indirect proof	*d.* An angle formed by extending one side of a triangle.
5. Postulate	*e.* Lines in the same plane that cannot meet.
6. Exterior ∠	*f.* Lines that meet at right angles.
7. Median	*g.* A line cutting two or more other lines.
8. Bisector	*h.* A line from a vertex of a triangle to the middle point of the opposite side.
9. Perpendicular lines	*i.* A construction admitted as possible without proof.
10. Parallel lines	*j.* A figure that cuts another figure in two equal parts.

REASONING TEST (*10 min.*)

If the conclusion is true, copy those statements from the hypothesis used in proving it, if false, write F.



1. If $\angle r = 70°$, $AB = CD$, $\angle s = 110°$ and EF is a st. line, then $AB \parallel CD$.

2. If $\angle p = \angle r$, $\angle q =$ supp. of $\angle y$ and $EF = CD$, then $AB \parallel CD$.

3. If $\angle w = 82°$, $AB \parallel CD$ and $\angle y = 82°$, then $\angle s = 98°$.

4. If $FN = GM$, $FG = HK$ and $\angle N = \angle M$, then $NH \parallel MK$.

5. If $NH \perp FL$, $FN \parallel GM$, $NH \parallel MK$ and $NH = MK$, then $MK \perp FL$.

COMPLETION TEST (*10 min.*)

Write the numbers of the questions on your answer paper and after each the one word that is omitted.

1. If two isosceles triangles have a common base, the line determined by their vertices is . . . to the base.

2. An exterior angle of a triangle is greater than a . . . interior angle.

3. The bisectors of a pair of corresponding angles of parallel lines are . . . to each other.

4. If two lines are cut by a transversal so that a pair of corresponding angles are equal, the interior angles on the same side of the transversal are . . . .

5. A line parallel to the base of an isosceles triangle cuts off an . . . triangle.

MULTIPLE-CHOICE TEST (*10 min.*)

From the answers given, select that one that will make the statement true.

1. If an obtuse angle and an acute angle have their sides respectively parallel, the two angles are, (*a*) equal; (*b*) rt. $\angle$s; (*c*) adj. $\angle$s; (*d*) supp. $\angle$s.

2. In the first of the theorems in which lines are proved parallel, the angles given equal are, (*a*) corr. $\angle$s; (*b*) int. $\angle$s on same side of transversal; (*c*) rt. $\angle$s; (*d*) alt. int. $\angle$s.

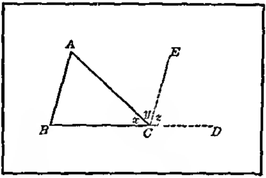
3. Lines perpendicular to the same line are, (*a*) equal; (*b*) parallel; (*c*) perpendicular; (*d*) form a triangle.

4. The angles of the letter N are (*a*) corr. $\angle$s; (*b*) supp. $\angle$s, (*c*) alt. int. $\angle$s; (*d*) obtuse $\angle$s.

5. If the opposite sides of a quadrilateral are parallel, a diagonal (*a*) bisects the $\angle$s; (*b*) makes 2 $\triangle$s $\cong$; (*c*) makes int. $\angle$s supp.; (*d*) equals the longer side.

Proposition 16

★ **86.** *The sum of the angles of a triangle is a straight angle.*



Given: $\triangle ABC$.

To prove: $\angle A + \angle B + \angle x$ equals a straight angle.

Proof: Statements	Reasons
1. Extend BC to D.	1. Post. 2.
2. Draw $CE \parallel BA$.	2. § 77.
3. $\angle x + \angle y + \angle z = 1$ st. angle.	3. Ax. 7.
4. $\angle y = \angle A$.	4. § 79.
5. $\angle z = \angle B$.	5. § 80.
6. $\angle x + \angle A + \angle B = 1$ st. angle.	6. Subst.

87. Corollary 1. *An exterior angle of a triangle equals the sum of the two remote interior angles.*

88. Corollary 2. *Each angle of an equilateral triangle equals* $60°$.

89. Corollary 3. *If two triangles have two angles of one equal respectively to two angles of the other, their third angles are equal.*

Class Exercises

1. Construct an angle of 60°, 30°, 120°.

2. Construct an angle of 150°, 105°, 75°.

3. Trisect a right angle.

4. Trisect an angle of 45°.

5. Construct a triangle whose base is 3 in. and whose base angles are 120° and 30° respectively.

6. Find the number of degrees in each angle of an isosceles right triangle.

7. The acute angles of a right triangle are complementary.

8. Given two angles of a triangle, construct the third angle.

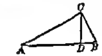
9. Given the vertex angle of an isosceles triangle, construct a base angle.

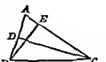
10. In $\triangle ABC$, find $\angle C$ if:

 (a) $A = 38°$ and $B = 74°$.
 (b) $A = B = C$.
 (c) $B = 48°$ and $A = C$.
 (d) $C = A + B$.
 (e) $A = m°$ and $B = n°$.
 (f) $C + A = 125°$ and $C + B = 130°$.

11. If the vertex angle of an isosceles triangle is 40°, find the exterior angle made by producing the base.

12. If CD is the altitude on the hypotenuse of right $\triangle ABC$, then $\angle ACD = \angle B$.





3. If BE and CD are two altitudes
lel; (c)?, prove that $\angle ABE = \angle ACD$.

4. The
(c) alt. int. um of two angles of a triangle equals the third angle,
5. If the opposite triangle.
13. isects the $\triangle$; (a triangle are 30° and 60°, what angle is
of $\triangle ABC$ nals the longer ?

16. If two angles of a triangle are equal, the bisector of the third angle bisects the triangle.

17. Find the angle formed by the bisectors of two angles of an equilateral triangle.

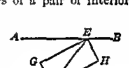
18. A triangle cannot have more than one right angle or more than one obtuse angle.

19. If a line bisects the legs of an isosceles triangle, it is parallel to the base.

20. If a line cuts off $\frac{1}{n}$th of the legs of an isosceles triangle, it is parallel to the base.

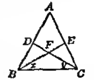
Optional Exercises

21. If two lines are parallel, the bisectors of a pair of interior angles on the same side of the transversal are perpendicular to each other.



22. If two parallel lines are cut by a transversal, the bisectors of the four interior angles form a rectangle. (A rectangle is a four-sided polygon all of whose angles are right angles.)

23. If a leg of an isosceles triangle is extended its own length through the vertex, the line joining its end to the nearer end of the base is perpendicular to the base.



24. The vertex $\angle A$ of an isosceles $\triangle ABC$ is 40°. If $BE \perp AC$, find $\angle EBC$.

25. Using the same hypothesis, if CD is also perpendicular to AB, find $\angle BFC$.

26. If $\angle A = 50°$, $\angle ABC = 70°$, $\angle ABF = m°$, and $\angle ACF = n°$, find all the angles of $\triangle BFC$.

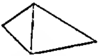
27. Given the same hypothesis, if $\angle BFC$ is a right angle, find the relation between m and n.

28. If, in the hypothesis of Ex. 26, BF and CF produced were perpendicular to AC and AB respectively, what values would m and n then have? Also find the number of degrees in each of the angles of $\triangle BFC$.

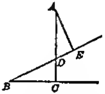
29. If a median of a triangle is half the side to which it is drawn, the triangle is a right triangle.

30. If the base of a triangle is extended in both directions, the sum of the exterior angles exceeds the vertex angle by a straight angle.

31. How many straight angles are there in the sum of the angles of a quadrilateral?



Ex. 31.



Ex. 35.

32. How many straight angles are there in the sum of the angles of a pentagon (5 sides)?

33. How many straight angles are there in the sum of the angles of a hexagon (6 sides)?

34. How does the number of straight angles in each of these figures compare with the number of sides?

35. *If the sides of $\angle A$ are perpendicular respectively to the sides of $\angle B$, prove $\angle A = \angle B$.*

36. If the bisectors of two angles of a triangle meet, they form an obtuse angle.

Algebraic Exercises

In $\triangle ABC$, find the number of degrees in each angle, if:

37. $\angle B$ is twice $\angle A$, and $\angle C$ is $20°$ more than $\angle A$.

38. $\angle B$ is three times $\angle A$, and $\angle C$ is twice $\angle B$.

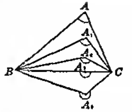
39. $\angle B$ exceeds $\angle A$ by $5°$, and $\angle C$ exceeds $\angle B$ by $5°$.

40. The sum of $\angle A$ and $\angle B$ is $130°$, and four times $\angle A$ exceeds five times $\angle B$ by $13°$.

41. $\angle B$ is twice $\angle A$, and $\angle C$ equals $\frac{1}{2}$ the sum of $\angle A$ and $\angle B$.

Honor Work

42. In $\triangle ABC$, what is the sum of the $\measuredangle A+B+C$?

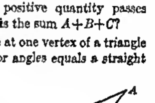


43. As A moves down toward BC to the positions $A_1, A_2, \ldots$, what change is taking place in the size of $\angle A$? Of $\angle B$? Of $\angle C$? How does this affect the sum $A+B+C$?

44. When A reaches the position A_3 on BC, what is the value of $\angle A$? Of $\angle B$? Of $\angle C$? What is their sum?

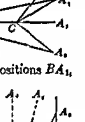
45. If A continues down across BC, what can you say about the size of $\angle A$? *As A moved toward BC, $\angle B$ decreased*, and when A reached A_3, $\angle B$ became 0. If $\angle B$ continues in the same direction, what does it then become? When a positive quantity passes through 0, what does it become? What is the sum $A+B+C$?

46. Prove that the angle on the outside at one vertex of a triangle minus the sum of the two remote interior angles equals a straight angle.



47. If point A moved toward BC extended, trace the changes in the size of the *three angles, (c) as it approaches* BC; (*b*) as it reaches BC produced at A_2; *(c) as it crosses BC to A_3.* What is the sum $A+B+C$ in each position?

48. In the figure, BA is rotating around B to the positions BA_1, $BA_2, \ldots$ Describe the change taking place in the size of $\measuredangle A$, B and C. How does the amount that A changes compare with the amount B changes? What effect has this on the sum $A+B+C$?

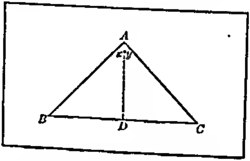


49. As BA becomes nearly parallel with CA, what can you say about the size of $\angle A$? About the sum $B+C$? When BA becomes parallel to CA, what is the value of $\angle A$? Of $\measuredangle B+C$? State a theorem in proof of the latter. What is the sum $A+B+C$?

50. If $\angle C$ is a right angle, what does $\angle B$ approach as A moves *up to a great distance? When BA becomes parallel to CA, what value has $\angle A$? $\angle B$?* What is the relation of BA to BC? What is the sum $A+B+C$?

Proposition 17

*91. *If two angles of a triangle are equal, the sides opposite them are equal.*



Given: $[\triangle ABC]$; $\angle B = \angle C$.
To prove: $AC = AB$.

Proof:

Statements	Reasons
1. Construct AD bisecting $\angle A$.	1. § 65.
2. In $\triangle ABD$ and ACD, $AD = AD$.	2. Iden.
3. $\angle x = \angle y$.	3. Const.
4. $\angle B = \angle C$.	4. Hyp.
5. $\triangle ABD \cong \triangle ACD$.	5. § 90.
6. $AB = AC$.	6. § 22.

92. Corollary 1. *An equiangular triangle is equilateral.*

93. Corollary 2. *The legs of a right triangle are equal if one of its acute angles equals 45°.*

Method of attack. *Lines can often be proved equal* by showing that they are sides of a triangle opposite equal angles.

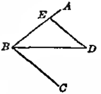
What method have you generally used for proving lines equal? Can lines be proved equal by means of Propositions 5 or 6? What axioms might also be used?

CLASS EXERCISES

1. If two angles of a triangle are 70° and 40°, the triangle is isosceles.

2. If the exterior angles at the base of a triangle are equal, the triangle is isosceles.

3. If BD is the bisector of $\angle ABC$ and $DE \parallel BC$, then $\triangle BDE$ is isosceles.



Ex. 3.

4. If any angle of an isosceles triangle is 60°, the triangle is equilateral. (2 cases.)

5. DE is parallel to the base BC of isosceles $\triangle ABC$, and ends in the other two sides. Prove that $\triangle ADE$ is isosceles.

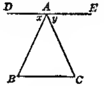
6. The legs JH and JI of isosceles $\triangle JHI$ are extended through the vertex J to G and F respectively. If $FG \parallel HI$, then $\triangle JFG$ is isosceles.

7. If the bisector of an exterior angle of a triangle is parallel to a side, the triangle is isosceles.

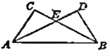
8. A line parallel to a leg of an isosceles triangle cuts off another isosceles triangle.

9. The bisectors of the base angles of an isosceles triangle form, with the base, another isosceles triangle.

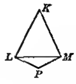
10. If $DE \parallel BC$ and $\angle x = \angle y$, then $AB = AC$.



Ex. 10.



Ex. 12.



Ex. 13

11. If the exterior angle at the vertex of an isosceles triangle is 120°, the triangle is equilateral.

12. If $AD = BC$ and $AC = BD$, prove that $AE = EB$.

13. In $\triangle KLM$, $KL = KM$, $LP \perp KL$, and $PM \perp KM$. Prove that $\triangle PLM$ is isosceles.

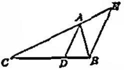
Optional Exercises

14. If the bisectors of two angles of a triangle form with the included side an isosceles triangle, the original triangle is also isosceles.

15. In isosceles $\triangle ABC$, CD and EB, the altitudes on the legs, meet at F. Prove that $\triangle BFC$ is isosceles.

16. An isosceles right triangle is divided by the altitude on the hypotenuse into two isosceles right triangles.

17. If, in quadrilateral $QRST$, $QR = RS$ and $\angle Q = \angle S$, then $TQ = TS$.



18. If AD bisects $\angle BAC$ of $\triangle ABC$, and $BE \parallel AD$, then $EA = AB$.

19. If $EA = AB$ and AD bisects $\angle BAC$, then $EB \parallel AD$.

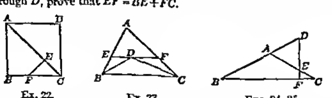
20. If, from a point in the base of an isosceles triangle, parallels to the legs are drawn, the perimeter of the quadrilateral formed equals the sum of the legs of the triangle.

21. A perpendicular to a diagonal of a square cuts off an isosceles triangle.

Honor Work

22. If in the square $ABCD$, $AE = AB$ and $EF \perp AC$, then $BF = EF = EC$.

23. If BD and CD bisect $\angle B$ and $\angle C$ respectively, and $EF \parallel BC$ through D, prove that $EF = BE + FC$.



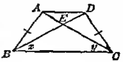
Ex. 22. Ex. 23. Exs. 24, 25.

24. If $\triangle ABC$ is isosceles, and $DF \perp$ base BC, then $\triangle ADE$ is isosceles.

25. If the vertex $\angle A$ of isosceles $\triangle ABC$ is twice the sum of the base angles, and $DEF \perp BC$, then $\triangle ADE$ is equilateral.

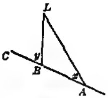
26. If the vertex angle of an isosceles triangle is a right angle, what is the ratio of its base to its altitude?

27. The bisectors of the equal angles of an isosceles triangle meet the opposite sides at D and E. *Prove that* DE *is parallel to the base.*

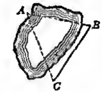


28. If in a quadrilateral the diagonals are equal, and also one pair of opposite sides are equal, prove that two of the triangles, into which the quadrilateral is divided by the diagonals, are isosceles.

Applied Problems

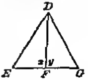


29. To find his distance from a lighthouse L, a ship captain measures $\angle x$, which he finds to be 33°. After sailing six miles to B, he observes that $\angle y$ is 66°. Prove that he is now just six miles from the lighthouse.



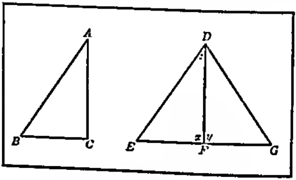
30. To find the distance across a pond from B to A, John measures $\angle B$ equal to 70°, and walks in the direction BC until he arrives at a point C, where $\angle ACB$ equals 55°. He then finds that BC is 210 ft. Find the distance from A to B.

Investigation Problem. The two triangles in the figure are right triangles having a leg DF in common, and the hypotenuse DE of one equal to the hypotenuse DG of the other. Do they appear to be congruent? How many parts of one are there that we know are equal to parts of the other? Using any proposition that we have had, can we prove that triangles having these parts are congruent? Would it help us to know that $\angle E$ equaled $\angle G$? If we knew that $DEFG$ was a triangle, could we prove $\angle E$ equal to $\angle G$? Is $DEFG$ a triangle? Is EFG a straight line? Why? If the two triangles were not together as in the above figure, what could we do?



Proposition 18

* 94. *Two right triangles are congruent, if the hypotenuse and a leg of one equal the hypotenuse and a leg of the other.*



Given: [$\triangle ABC$ and DEF]; $\angle C$ and $\angle F$ right angles, $AB = DE$, and $AC = DF$.

To prove: $\triangle ABC \cong \triangle DEF$.

Proof:

Statements	Reasons
1. Place $\triangle ABC$ so that point C falls on point F, CA runs along FD, and B takes the position G on the opposite side of DF from E.	1. Ax. 10.
2. Point A falls on point D.	2. $AC = DF$, by hyp.
3. $\angle x$ and $\angle y$ are rt. angles.	3. Hyp.
4. $\angle EFG$ is a st. angle.	4. Two rt. angles = a st. angle.
5. EFG is a st. line.	5. § 10.
6. $DE = AB$ or DG.	6. Hyp.
7. $\angle E = \angle G$.	7. § 55.
8. $\angle x = \angle y$.	8. § 33.
9. $\triangle DEF \cong \triangle DFG$ or $\triangle ABC$.	9. § 90.

95. Corollary. *If an angle of a right triangle is* 30°, *the side opposite is one-half the hypotenuse.*

If, in the diagram of Proposition 18, $\triangle DEF \cong \triangle DGF$ and $\angle z = 30°$, show that $\triangle DEG$ is equilateral, and therefore that $EF = \frac{1}{2}DE$.

96. The distance from a point to a line is the length of the perpendicular from the point to the line

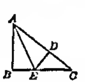
Exercises

1. If P is equally distant from A and B, and $PC \perp AB$, then PC bisects AB.

2. How many and what parts of two triangles must be known to be equal, in order that the triangles can be proved congruent?

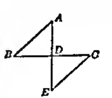
3. If the perpendiculars from the middle point of a side of a triangle on the other two sides are equal, the triangle is isosceles.

4. In right $\triangle ABC$, $AD = AB$, and $DE \perp$ hypotenuse AC, prove that AE bisects $\angle BAC$.



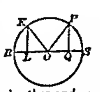
5. If the base BC of $\triangle ABC$ is trisected at D and E, and the perpendiculars to the other sides, DF and EC, are equal, $\triangle ABC$ is isosceles.

6. Equal oblique lines from a point on a perpendicular to a line cut off equal distances from the foot of the perpendicular.



7. If AE is the perpendicular bisector of BC, and $AB = CE$, show that $AB \parallel CE$.

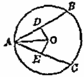
8. *A point equally distant from the sides of an angle is on the bisector of the angle.*



9. In a circle, a radius perpendicular to a chord bisects the chord.

10. In circle O, if KL and PQ are equal to each other and are perpendicular to diameter RS, then $LO = OQ$.

11. In square $ABCD$, E is a point on BC. If $EA = ED$, prove that E is the middle point of BC.



12. If AB and AC are equally distant from the center of circle O, then AO bisects $\angle BAC$.

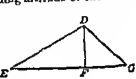
13. A triangle is isosceles if two of its altitudes are equal.

14. Two acute triangles are congruent, if two sides and the altitude on one of them in one triangle equal respectively two sides and the corresponding altitude of the other.

15. In $\triangle ABC$, $\angle A = 30°$, $\angle C = 90°$, and $AB = 12$. Find BC.

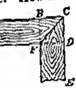
16. In $\triangle ABC$, $\angle A = 30°$, $\angle B = 60°$, and $BC = 7$. Find AB.

17. In $\triangle DEG$, $\angle E = 30°$, $\angle G = 45°$, $DF \perp EG$ and $ED = 12$. Find the length of FG.

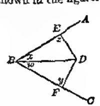


Applied Problems

18. A carpenter wishes to saw a board at a 45° angle. How can he lay off this angle?



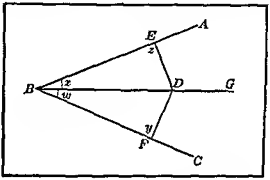
19. Prove that, if a carpenter wishes to cut two boards, whose widths are equal, so that they will fit together to form a right angle, as shown in the figure, he must cut each at 45°, for, if he cut them at any other angle, say 60° and 30°, the slanting parts would not fit together as shown in the figure.



Investigation Problem. D is a point on BD, the bisector of $\angle ABC$. DE and DF are its distances from AB and BC respectively. What is meant by the distance from a point to a line? How do these distances compare in length? Can you prove this? State the converse of the above. Does it appear to be true? What is the method of proving $\angle x$ equal to $\angle w$?

Proposition 19

* 97. *Any point on the bisector of an angle is equally distant from the sides of the angle.*



Given: BG bisects $\angle ABC$, D any point on BG, DE and DF the distances from D to AB and BC respectively.

To prove: $DE = DF$.

Proof: STATEMENTS	REASONS
1. In $\triangle BDE$ and BDF, $\angle x = \angle w$.	1. Hyp.
2. $BD = BD$.	2. Iden.
3. $\angle z$ and $\angle y$ are rt. angles.	3. § 96.
4. $\angle z = \angle y$.	4. § 33.
5. $\triangle BDE \cong \triangle BDF$.	5. § 90.
6. $DE = DF$.	6. § 22.

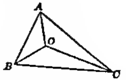
* 98. Converse. *Any point equally distant from the sides of an angle is on the bisector of the angle.*

Does this proposition furnish us with any new method of proving lines equal? Of proving angles equal?

If D moves along BG, will it always be as far from AB as from BC?

Exercises

1. If BO bisects $\angle B$ and CO bisects $\angle C$, then O is equally distant from the three sides.



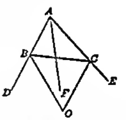
2. Using the hypothesis of Ex. 1, if AO is drawn, it will bisect $\angle A$.

3. Construct a point equally distant from all three sides of a given triangle.

4. A point not on the bisector of an angle is unequally distant from its sides.

5. A point unequally distant from the sides of an angle is not on the bisector of the angle.

6. On one side of a triangle, find a point equally distant from the other two sides.



7. If BO and CO are the bisectors of the exterior $\angle DBC$ and ECB respectively, prove that O lies on the bisector of $\angle A$.

8. Three indefinite lines intersect, forming a triangle. Construct four points, each equally distant from all three lines.

9. Is it possible to find a point equally distant from three sides of a quadrilateral? From all four sides? Explain.

A Self-Measuring Test

1. Give four methods of proving triangles congruent.

2. What extra method can be used to prove triangles congruent, if it is known that the triangles are right triangles?

3. How many methods of proving right triangles congruent do you know?

4. State two facts which you have learned about the exterior angle of a triangle.

5. Give three methods of proving lines equal.

6. Give seven methods of proving angles equal.

7. State two propositions about the isosceles triangle.

8. Define an *altitude* of a triangle; a *median*.

9. Define *parallel lines*.

10. Why is it that the first proposition about the isosceles triangle is placed among the congruent triangle propositions, while its converse comes so much later?

11. Why is the proposition regarding congruent triangles, given two angles and the side opposite one of them, not placed with the other three propositions about congruent triangles?

12. What is the sum of the angles of a triangle?

13. What method of proving angles equal depends on the sum of the angles of a triangle?

14. Which two congruent-triangle propositions depend on the sum of the angles of a triangle?

15. Give two methods of proving that an angle is a right angle or that two lines are perpendicular.

16. Give two methods of proving lines parallel.

17. If two angles of a triangle are of x degrees and y degrees respectively, how many degrees are there in the third angle?

18. Are two triangles congruent if three angles of one equal respectively three angles of the other?

19. What are *supplementary angles? Adjacent angles? Vertical angles?*

20. In a triangle, how many of the angles can be right angles? Obtuse angles? Acute angles?

21. If a right triangle has an angle of 30°, what relation of the sides is known?

22. What fact has been proved about the diagonals of a square? Of a rhombus?

Numerical Test (*10 min.*)

1. The bisectors of angles A and B of the equilateral triangle ABC meet at D. How many degrees are there in $\angle ADB$?

2. The angles of a triangle are in the ratio $3:4:5$. Find the number of degrees in the largest angle.

3. In right triangle ABC, the exterior angle at A contains 140°. Find the number of degrees in the acute angle B.

4. Two angles of a triangle are in the ratio 5 : 3 and their difference is 40°. Find the smallest angle of the triangle.

Completion Test (*10 min.*)

Write the numbers of the questions on your answer paper and after each the word that is omitted.

1. If two altitudes of a triangle are equal, then the triangle is . . . .
2. In the right triangle ABC, if $\angle B = 30°$ and $AC = 2$ in., then the hypotenuse AB equals . . . inches.
3. The distance from a point to a line is the length of the . . . from the point to the line.
4. If the sum of two angles of a triangle is greater than a right angle, the third angle is . . . than a right angle.
5. Any point equally distant from the sides of an angle is on the . . . .
6. If one angle of a right triangle is 45°, the legs are . . . .
7. A point equally distant from the ends of a segment is on the . . . .
8. If two angles of a triangle are 70° and 40°, the triangle must be an . . . triangle.

Multiple-Choice Test (*10 min.*)

From the answers given, select the one that will make the statement true.

1. If the sum of two angles of a triangle equals the third angle, the triangle is, (*a*) isosceles; (*b*) equilateral; (*c*) obtuse; (*d*) right.
2. If from any point in the bisector of an angle, a line is drawn parallel to one side of the angle, the triangle formed is (*a*) isosceles; (*b*) right; (*c*) equiangular; (*d*) congruent.
3. If two angles of a triangle are equal, (*a*) the third is equal; (*b*) the triangles are congruent; (*c*) the triangle is isosceles; (*d*) the triangle is a right triangle.
4. If two angles of a triangle are 65° and 50°, the triangle is, (*a*) right; (*b*) isosceles; (*c*) equiangular; (*d*) obtuse.

5. If an angle of a right triangle is 60°, (*a*) the adjacent leg is half the hypotenuse; (*b*) the triangle is equilateral; (*c*) the legs are equal; (*d*) the opposite leg is half the adjacent.

Reasoning Test *(10 min.)*

Give the final reason only for each of the following:

1. In $\triangle ABC$, $AB = AC$ and OB and OC are angle bisectors. Then $OB = OC$.

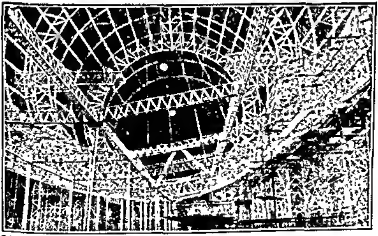
2. Lines from the middle point of a side of an equilateral triangle perpendicular to the other two sides cut off congruent triangles.

3. The bisector of an acute angle of a right triangle cuts the opposite leg in a point equally distant from the right angle and the hypotenuse.

4. A triangle is isosceles if the exterior angles at two of its vertices are equal.

Quadrilaterals

99. A trapezoid is a quadrilateral having two, and only two, sides parallel. It is isosceles if the non-parallel sides are equal. The parallel sides are called the **bases**.



Official Photograph, U. S. Navy

Part of the superstructure of the dirigible "Macon."
Notice the many geometrical forms in the girders themselves and in their arrangement.

3. In right triangle ABC, the exterior angle at A contains 140°. Find the number of degrees in the acute angle B.

4. Two angles of a triangle are in the ratio $5:3$ and their difference is 40°. Find the smallest angle of the triangle.

Completion Test (*10 min.*)

Write the numbers of the questions on your answer paper and after each the word that is omitted.

1. If two altitudes of a triangle are equal, then the triangle is . . . .
2. In the right triangle ABC, if $\angle B = 30°$ and $AC = 2$ in., then the hypotenuse AB equals . . . inches.
3. The distance from a point to a line is the length of the . . . from the point to the line.
4. If the sum of two angles of a triangle is greater than a right angle, the third angle is . . . than a right angle.
5. Any point equally distant from the sides of an angle is on the . . . .
6. If one angle of a right triangle is 45°, the legs are . . . .
7. A point equally distant from the ends of a segment is on the . . . .
8. If two angles of a triangle are 70° and 40°, the triangle must be an . . . triangle.

Multiple-Choice Test (*10 min.*)

From the answers given, select the one that will make the statement true.

1. If the sum of two angles of a triangle equals the third angle, the triangle is, (*a*) isosceles; (*b*) equilateral; (*c*) obtuse; (*d*) right.
2. If from any point in the bisector of an angle, a line is drawn parallel to one side of the angle, the triangle formed is (*a*) isosceles; (*b*) right; (*c*) equiangular; (*d*) congruent.
3. If two angles of a triangle are equal, (*a*) the third is equal; (*b*) the triangles are congruent; (*c*) the triangle is isosceles; (*d*) the triangle is a right triangle.
4. If two angles of a triangle are 65° and 50°, the triangle is, (*a*) right; (*b*) isosceles; (*c*) equiangular; (*d*) obtuse.

5. If an angle of a right triangle is 60°, (a) the adjacent leg is half the hypotenuse; (b) the triangle is equilateral; (c) the legs are equal; (d) the opposite leg is half the adjacent.

Reasoning Test (*10 min.*)

Give the final reason only for each of the following:

1. In $\triangle ABC$, $AB = AC$ and OB and OC are angle bisectors. Then $OB = OC$.

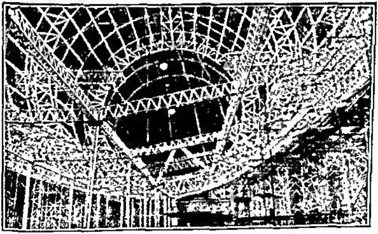
2. Lines from the middle point of a side of an equilateral triangle perpendicular to the other two sides cut off congruent triangles.

3. The bisector of an acute angle of a right triangle cuts the opposite leg in a point equally distant from the right angle and the hypotenuse.

4. A triangle is isosceles if the exterior angles at two of its vertices are equal.

Quadrilaterals

99. A trapezoid is a quadrilateral having two, and only two, sides parallel. It is isosceles if the non-parallel sides are equal. The parallel sides are called the bases.



Official Photograph, U. S. Navy

Part of the superstructure of the dirigible "Macon."
Notice the many geometrical forms in the girders themselves and in their arrangement.

100. A parallelogram (▱) is a quadrilateral having both pairs of opposite sides parallel.

101. A rectangle is a parallelogram, one of whose angles is a right angle.

It can be proved that:

(*a*) *All the angles of a rectangle are right angles*; and

(*b*) *Any quadrilateral is a rectangle if all of its angles are right angles.*

Ex. Prove (*a*) and (*b*).

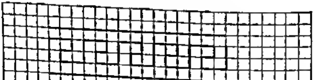
102. In a parallelogram, either pair of parallel sides may be called bases. Generally the side on which the parallelogram appears to rest is called the base.

103. An altitude of a trapezoid or parallelogram is a perpendicular between parallel sides.

The parallelogram has two different altitudes, one to each pair of parallel sides.

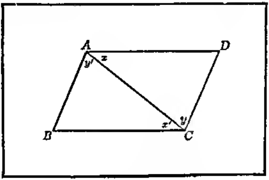
Drawing Exercise

Draw the following Greek designs:



PROPOSITION 20

104. *A diagonal of a parallelogram divides it into two congruent triangles.*



Given: $\square ABCD$; [diagonal AC].

To prove: $\triangle ABC \cong \triangle ACD$.

Proof: STATEMENTS	REASONS
1. In $\triangle ABC$ and ACD, $AC = AC$.	1. Iden.
2. $AD \parallel BC$ and $AB \parallel CD$.	2. § 100.
3. $\angle x = \angle x'$ and $\angle y = \angle y'$.	3. § 79.
4. $\triangle ABC \cong \triangle ACD$.	4. § 52.

105. Corollary 1. *The opposite sides of a parallelogram are equal and the opposite angles are equal.*

106. Corollary 2. *Segments of parallels intercepted between parallels are equal.*

107. A rhombus is a parallelogram having two adjacent sides equal.

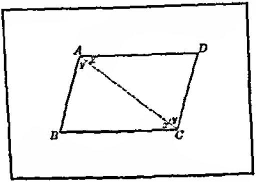
Then, from 105; *A rhombus is an equilateral parallelogram.*

108. A square is a rectangle having two adjacent sides equal.

Then, from 105; *A square is an equilateral rectangle.*

Proposition 21

★ 109. *A quadrilateral is a parallelogram if its opposite sides are equal.*



Given: [Quadrilateral $ABCD$]: $AB = CD$ and $AD = BC$.

To prove: $ABCD$ is a parallelogram.

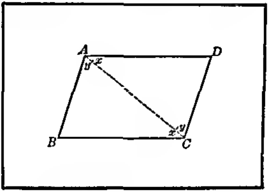
Proof: Statements	Reasons
1. Construct AC.	1. Post. 1.
2. In $\triangle ABC$ and ACD, $AC = AC$.	2. Iden.
3. $AB = CD$ and $AD = BC$.	3. Hyp.
4. $\triangle ABC \cong \triangle ACD$.	4. § 57.
5. $\angle x = \angle x'$ and $\angle y = \angle y'$.	5. § 22.
6. $AD \parallel BC$ and $AB \parallel DC$.	6. § 72.
7. $ABCD$ is a $\square$.	7. § 100.

Classify quadrilaterals according to the number of pairs of parallel sides. How many pairs of opposite sides has the quadrilateral? How many of these may be parallel? How many are parallel in the parallelogram? In the trapezoid? Are there other quadrilaterals besides these? How do they differ from these?

What do we call a parallelogram which has right angles? Are there other parallelograms? Name the equilateral parallelogram. Is the square a special rhombus?

PROPOSITION 22

***110.** *A quadrilateral is a parallelogram, if two sides are equal and parallel.*



Given: [Quadrilateral $ABCD$]; $AD = BC$ and $AD \parallel BC$.

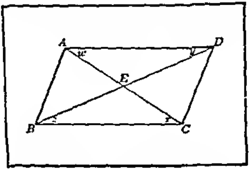
To prove: $ABCD$ is a parallelogram.

Proof: STATEMENTS	REASONS
1. Construct AC.	1. Post. 1.
2. In $\triangle ABC$ and ACD, $AD = BC$.	2. Hyp.
3. $AC = AC$.	3. Iden.
4. $AD \parallel BC$.	4. Hyp.
5. $\angle x = \angle x'$.	5. § 79.
6. $\triangle ABC \cong \triangle ACD$.	6. § 50.
7. $\angle y = \angle y'$.	7. § 22.
8. $AB \parallel CD$.	8. § 72.
9. $ABCD$ is a $\square$.	9. § 100.

Investigation Problem. In parallelogram $ABCD$, not a rectangle, draw both diagonals, AC and BD, intersecting at E. Does AC equal BD? Does AE equal EC? Does BE equal ED? Try to prove your conclusion without further help. What method of proving lines equal can you use here? What properties of the parallelogram will help you?

Proposition 23

111. *The diagonals of a parallelogram bisect each other.*



Given: ▱$ABCD$; [with AC and BD intersecting at E].

To prove: $AE = EC$, and $BE = ED$.

Proof:

Statements	Reasons
1. In $\triangle AED$ and BEC, $AD = BC$.	1. § 105.
2. $AD \parallel BC$.	2. § 100.
3. $\angle w = \angle x$, and $\angle y = \angle z$.	3. § 79.
4. $\triangle AED \cong \triangle BEC$.	4. § 52.
5. $AE = EC$ and $BE = ED$.	5. § 22.

112. Converse. *A quadrilateral is a parallelogram if its diagonals bisect each other.*

113. Method of attack. *Lines may be proved parallel or equal* by showing that they are opposite sides of a parallelogram.

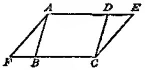
Class Exercises

1. The opposite angles of a parallelogram are equal.
2. Parallel lines are everywhere the same distance apart.

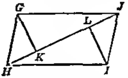
3. The line joining the middle points of two opposite sides of a parallelogram is parallel to the other two sides.

4. Two lines are parallel if two points on one of them are equally distant from the other.

5. If two opposite sides, AD and CB, of $\square ABCD$ are extended equal lengths in opposite directions to E and F respectively, the figure $AFCE$ is a parallelogram.



Ex. 5.



Ex. 7.

6. Perpendiculars to one side of a parallelogram from the opposite vertices are equal.

7. In a parallelogram, perpendiculars to a diagonal from the opposite vertices are equal.

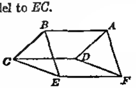
8. The diagonals of a rectangle are equal.

9. If the diagonals of a parallelogram are equal, the figure is a rectangle.

10. A line through the middle point of a diagonal of a parallelogram and ending in a pair of opposite sides, is bisected hy the diagonal.

11. If the diagonal of a parallelogram bisects one angle, it also bisects the other angle.

12. If E and F are the middle points of the sides AD and BC respectively of $\square ABCD$, then AF is parallel to EC.



13. If a parallelogram is not equilateral, the bisectors of the opposite angles are parallel.

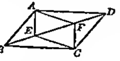
14. $ABCD$ and $ABEF$ are two parallelograms. Prove that CE equals DF.

15. If an angle of a rhombus is 50°, find the angles of the triangles formed hy drawing the diagonals.

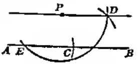
Optional Exercises

16. If the middle points of the four halves of the diagonals of a parallelogram are joined in order, another parallelogram will be formed.

17. If BE and DF are equal distances taken on the diagonal BD of $\square ABCD$, then $AECF$ is a parallelogram.



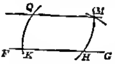
Ex. 17.



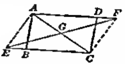
Ex. 18.

18. To construct a line through P parallel to AB, take P as center and construct an arc ED cutting AB at E. With E as center and the same radius, cut AB at C, and with C as center and the same radius, cut arc ED at D. Prove PD parallel to AB.

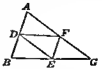
19. To construct a line through Q parallel to FG, take Q as center and, with any convenient radius, cut FG at H. With H as center and the same radius, cut FG at K. Then with H as center and QK as radius, intersect the first arc at M. Prove that QM is parallel to FG.



Ex. 19.



Ex. 20.



Ex. 21.

20. In $\square ABCD$, AD and CB are produced equal lengths to F and E respectively. Prove that EF bisects AC.

21. If DF, FE, and ED are parallel respectively to BC, AB and AC, then E, D, and F are the middle points of the sides.

22. The bisectors of the angles of a parallelogram which is not equilateral form a rectangle.

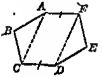
23. In a rectangle not a square, the bisectors of the angles form a square.

HONOR WORK

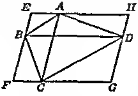
24. The chords which join the ends of any two diameters of a circle form a rectangle.

25. If the opposite sides of a hexagon (six-sided polygon) are parallel and two opposite sides are equal, the other opposite sides are equal.

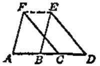
26. If the opposite sides of a hexagon are equal and parallel, the three diagonals between opposite vertices meet in a point.



Ex. 25.



Ex. 27.



Ex. 28.

27. If through the vertices of any quadrilateral lines are drawn parallel to its diagonals, the lines form a parallelogram which is twice as large as the original quadrilateral.

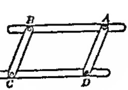
28. If $AB = CD$, $AF = BE$, and $AF \parallel BE$, prove, without using triangles, that FC is parallel to ED.

29. The medians of a triangle cannot bisect each other.

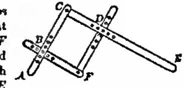
30. Is it possible to change the size of an angle of a parallelogram without changing the length of any of the sides? Is it possible to change the lengths of a *pair of opposite* sides without changing the lengths of the other pair or the size of any angle? Are the angles, then, dependent on the length of the sides? Is the same thing true of the triangle?

APPLIED PROBLEMS

31. Draughtsmen sometimes use rulers hinged at A, B, C, and D, as shown in the figure, having $AB = CD$ and $AD = BC$. If the ruler CD remains stationary, *show that a series of* parallel lines can be drawn along AB.

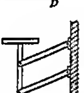


32. The pantograph is an instrument used for making a different-sized copy of a map or drawing. It is made of four strips AC, CE, BF, and DF, hinged at the points B, C, D, and F. BF is equal in length to CD, and BC to FD. Show that, although the distance between A and E may be changed, BF will always remain parallel to CE.

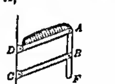


33. To find the distance AB, $\angle A$ is made 110° and $\angle B$ 70°, and equal distances AC and BD are measured. Prove that AB is equal to CD.

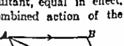
34. Explain how a movable shelf, like that in the figure, can be constructed so that it will always remain horizontal.



35. In the New York City subway, which is very crowded, there are seats, hinged at A, B, C, and D, which fold up against the wall when not in use. $AB = CD$, $BF = CE$, and $AD = BC$. Prove that, when the seat AD is down, the support AF will be vertical and the seat will be horizontal.



36. When two forces are pulling at the same point, but in different directions, the physicist finds a force, the resultant, equal in effect, both in amount and direction, to the combined action of the two forces. This greatly simplifies problems dealing with forces. He draws two lines, AB and AC, through the point A, in the direction of the forces. Then he chooses a suitable unit of length, and measures as many units on each line as there are pounds in the force represented by that line. Next he completes a $\square ABDC$ having these two lines as sides, and draws the diagonal AD. This diagonal gives the direction of the resultant, and its length gives the amount in pounds Co-



struct a parallelogram representing two forces, one of 2 lbs. and the other of 3 lbs., acting at an angle of 60°. By measuring, find the amount of the resultant.

37. Two forces, each 4 lbs., act at right angles on a point. Construct the parallelogram and measure the resultant.

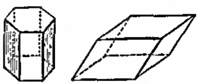
38. The same parallelogram method holds true also for velocities (speeds). If a man is swimming 2 mi. an hour across a river whose current is flowing 4 mi. an hour, find his actual direction and speed.

39. *A man is walking 5 ft. per second directly across the deck* of a boat which is traveling 12 ft. per second. Find his actual speed and direction.

40. An airplane which can go at the rate of 80 mi. an hour in still air, is traveling east. Determine how far it will travel in 1 hr., if the wind is blowing 40 mi. an hour (*a*) directly east; (*b*) directly west; (*c*) directly north; (*d*) 30° east of north.

SPACE GEOMETRY (*Optional*)

The prism. The prism is a solid like that shown here. The top and bottom are parallel planes and the edges between them are all parallel lines. These two parallel planes are called bases. The other faces are called lateral faces.



Prism *Parallelepiped*

The parallelepiped is a prism whose bases are parallelograms.

EXERCISES

1. Name several objects that are parallelepipeds; prisms.

2. How many faces has a parallelepiped? How many edges? How many vertices?

3. How many edges of a parallelepiped meet at each vertex?

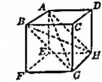
4. How many edges of a parallelepiped are necessarily the same length? Prove that they are equal.

5. How many edges different in length has the parallelepiped?

6. Three edges of a parallelepiped are 4, 6 and 9. What is the sum of the lengths of all edges?

7. Prove that the lateral faces of a prism are parallelograms and that the lateral edges are equal.

8. If a prism has triangular bases, prove that these bases are congruent.



9. A cube is a parallelepiped whose faces are all squares. A cube has how many faces meeting at each vertex? How many right angles at each vertex? How many diagonals?

10. If an edge of a cube is 10, find the sum of its edges.

11. Prove that a diagonal of one face of a cube equals a diagonal of any other face.

12. Prove that two diagonals of a cube AG and BH are equal.

13. Prove that two diagonals of a cube bisect each other. (Show that the plane $ABGH$ is a parallelogram.)

A Self-Measuring Test

1. Give three methods of proving that a quadrilateral is a parallelogram.

2. Give four methods of proving lines parallel.

3. Which method of proving lines parallel depends on quadrilaterals?

4. Give four methods of proving lines equal.

5. What two methods of proving lines equal depend on the parallelogram?

6. Define a *parallelogram*.

7. Name two special kinds of parallelograms.

8. Define a *square;* a *rhombus.*

9. How does a trapezoid differ from a parallelogram?

10. How does a rectangle differ from other parallelograms?

11. Are all rectangles parallelograms?

12. Are all squares rectangles? Are all rectangles squares?

13. Give five methods of proving right triangles congruent.

14. If a triangle has angles of 30, 60, and 90 degrees respectively, what relation of the sides is known?

15. Define *supplementary angles; adjacent angles; vertical angles;* a *right angle; quadrilateral.*

16. Is the converse of a proposition always true?

17. Define the *distance* from a point to a line.

18. How many different altitudes has a triangle? A parallelogram? A trapezoid?

19. State a proposition which is proved by the indirect method.

20. Name two purposes for either of which we might prove that a quadrilateral is a parallelogram.

21. Give a method of proving one line perpendicular to another.

22. Give two propositions on which the proof of each of the following propositions depends:

(*a*) The opposite sides of a parallelogram are equal.

(*b*) The sum of the angles of a triangle is a straight angle.

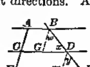
(*c*) Two right triangles are congruent, if the hypotenuse . . .

(*d*) Two triangles are congruent, if the three sides of one . . .

(*e*) The diagonals of a parallelogram bisect each other.

(*f*) A quadrilateral is a parallelogram: if its opposite sides are equal; if two sides are equal and parallel.

Investigation Problem. Examine a sheet of ruled writing paper. Are the lines parallel? Do they cut off equal lengths on the edge of the page? Draw several transversals in different directions. Are the lengths cut off by the parallel lines the same on one of them as on another? Are all the segments on any one transversal equal to each other? In the figure, if AB, CD, and EF are parallel, and AC equals CE, do you think that BD equals DF? Try drawing BG and DH both parallel to AE. To prove these triangles congruent, what sides must you prove equal? Compare BG with AC; DH with CE; BG with DH. Can you now prove BD equal to DF? If you give up, read Proposition 24.



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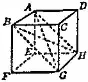

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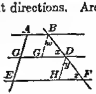
(*d*) Two triangles are congruent, if the three sides of one . . .

(*e*) The diagonals of a parallelogram bisect each other.

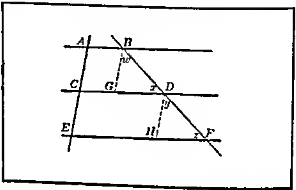
(*f*) A quadrilateral is a parallelogram: if its opposite sides are equal; if two sides are equal and parallel.

Investigation Problem. Examine a sheet of ruled writing paper. Are the lines parallel? Do they cut off equal lengths on the edge of the page? Draw several transversals in different directions. Are the lengths cut off by the parallel lines the same on one of them as on another? Are all the segments on any one transversal equal to each other? In the figure, if AB, CD, and EF are parallel, and AC equals CE, do you think that BD equals DF? Try drawing BG and DH both parallel to AE. To prove these triangles congruent, what sides must you prove equal? Compare BG with AC; DH with CE; BG with DH. Can you now prove BD equal to DF? If you give up, read Proposition 24.



Proposition 24

★ **114.** *If three or more parallel lines cut off equal lengths on one transversal, they cut off equal lengths on every transversal.*

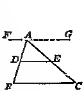


Given: $AB \parallel CD \parallel EF$ and $AC = CE$; [and transversal BF].

To prove: $BD = DF$.

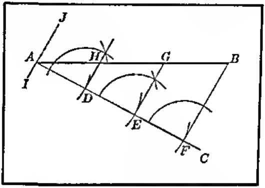
Proof: Statements	Reasons
1. Construct $BG \parallel AE$ and $DH \parallel AE$.	1. § 77.
2. $BG \parallel DH$.	2. § 82.
3. $AB \parallel CD \parallel EF$.	3. Hyp.
4. $AC = BG$ and $CE = DH$.	4. § 105.
5. $AC = CE$.	5. Hyp.
6. In $\triangle BGD$ and DHF, $BG = DH$.	6. Ax. 2.
7. $\angle w = \angle y$ and $\angle x = \angle z$.	7. § 80.
8. $\triangle BGD \cong \triangle DHF$.	8. § 90.
9. $BD = DF$.	9. § 22.

115. Corollary. *A line bisecting one side of a triangle and parallel to a second side, bisects the third side.*



Proposition 25

116. *A given line segment can be divided into any number of equal parts.*



To prove: AB can be divided into any number of equal parts.

Given: Line segment AB.

Construction: Statements	Reasons
1. Draw AC to any point C not on AB.	1. Post. 1.
2. Construct $AD=DE=EF$.	2. Post. 3.
3. Construct FB.	3. Post. 1.
4. Construct EG and DH parallel to FB.	4. § 77.

Then $AH=HG=GB$.

Proof:

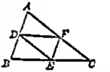
1. $AD=DE=EF$.	1. Const.
2. $HD \| GE \| BF$.	2. Const.
3. Through A, draw $IJ \| FB$.	3. § 77.
4. $AH=HG=GB$.	4. § 114.
5. Similarly, AB can be divided into any number of equal parts.	

117. Corollary. *A line bisecting two sides of a triangle is parallel to the third side.*

Class Exercises

1. Divide a line into five equal parts.
2. Divide a line into four equal parts by two different methods.
3. Find $\frac{3}{7}$ of a given line.
4. Construct an equilateral triangle, given the perimeter.
5. Construct a triangle whose sides are respectively $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{2}$ of a given line.

Optional Exercises



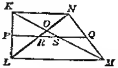
6. If E is the middle point of BC, $ED \parallel CA$, and $EF \parallel BA$, then F and D are the middle points of AC and AB respectively.

7. Using the hypothesis of Ex. 6, prove that a line through D parallel to BC passes through F.

8. Using the above hypothesis, prove that these lines cut $\triangle ABC$ into four congruent triangles.

9. A line cutting off one-third of one side of a triangle, and parallel to a second side, cuts off one-third of the remaining side.

10. A line cutting off $\frac{1}{n}$th of one side of a triangle, where n is a whole number, and parallel to a second side, cuts off $\frac{1}{n}$th of the third side.



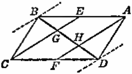
11. The perpendicular bisector of either leg of a right triangle bisects the hypotenuse.

12. A line bisecting one leg of a trapezoid and parallel to the bases bisects the other leg.

13. A line bisecting one leg of a trapezoid and parallel to the bases, bisects the diagonals.

Honor Work

14. In the parallelogram *ABCD*, *E* and *F* are the middle points of the sides *AB* and *CD* respectively; prove that the lines *AF* and *CE* trisect the diagonal *BD*.

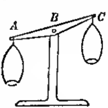


15. If three parallel lines cut off unequal segments on one transversal, the second segment being three times the first, prove that they will cut off unequal segments on any other transversal. How will the lengths of the segments on the second transversal compare? Prove your conclusion.

Applied Problems

16. By means of a ruled sheet of writing paper, divide a given line into three equal parts. How short a line can you divide by this method?

17. A carpenter, wishing to cut a board into three strips, equal in width, placed his square, as shown, near one end of the board, with the 0 and 24-in. divisions at the edges, and made marks at the 8- and 16-in. divisions. He repeated this at the other end of the board, and drew lines between the two sets of marks. Explain why this should divide the board into equal strips.



18. The operation of the balance depends on having the two pans equally distant from the center post. Prove that, if the lengths *AB* and *BC* are equal and the post is vertical, the pans will always hang at equal distances from the post.

Class Exercises

1. Divide a line into five equal parts.

2. Divide a line into four equal parts by two different methods.

3. Find $\frac{3}{5}$ of a given line.

4. Construct an equilateral triangle, given the perimeter.

5. Construct a triangle whose sides are respectively $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of a given line.

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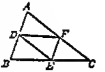

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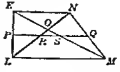

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Honor Work

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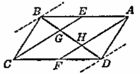

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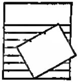

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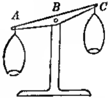

Class Exercises

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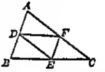
2. Divide a line into four equal parts by two different methods.
3. Find $\frac{2}{3}$ of a given line.
4. Construct an equilateral triangle, given the perimeter.
5. Construct a triangle whose sides are respectively $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of a given line.

Optional Exercises

6. If E is the middle point of BC, $ED \parallel CA$, and $EF \parallel BA$, then F and D are the middle points of AC and AB respectively.



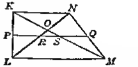
7. Using the hypothesis of Ex. 6, prove that a line through D parallel to BC passes through F.

8. Using the above hypothesis, prove that these lines cut $\triangle ABC$ into four congruent triangles.

9. A line cutting off one-third of one side of a triangle, and parallel to a second side, cuts off one-third of the remaining side.

10. A line cutting off $\frac{1}{n}$th of one side of a triangle, where n is a whole number, and parallel to a second side, cuts off $\frac{1}{n}$th of the third side.

11. The perpendicular bisector of either leg of a right triangle bisects the hypotenuse.



12. A line bisecting one leg of a trapezoid and parallel to the bases bisects the other leg.

13. A line bisecting one leg of a trapezoid and parallel to the bases, bisects the diagonals.

4. The sum of the angles of a triangle is a straight angle.

5. If two angles of a triangle are equal, the sides opposite . . . .

6. A quadrilateral is a parallelogram if two sides are equal and parallel.

MATCHING TEST (*10 min.*)

Write the numbers 1 to 10 in a column. After each write the letter of the phrase that is a correct definition of the word following the number.

1. Converse	*a.* A quadrilateral having two and only two sides parallel.
2. Rhombus	*b.* An equilateral rectangle.
3. Hypothesis	*c.* A quadrilateral having both pairs of opposite sides parallel.
4. Trapezoid	*d.* The parallel sides of a trapezoid.
5. Parallelogram	*e.* A parallelogram having right angles.
6. Altitude	*f.* A theorem in which the hypothesis and conclusion of another theorem are interchanged.
7. Indirect proof	*g.* A proof in which we show that the other possibilities are not true.
8. Rectangle	*h.* An equilateral parallelogram.
9. Square	*i.* A perpendicular from a vertex of a triangle to the opposite side.
10. Bases	*j.* The part of a theorem that is given.

NUMERICAL TEST (*10 min.*)

1. In $\square ABCD$, $\angle A = 73°$. Find $\angle B$.

2. AC is the diagonal of rhombus $ABCD$. If $\angle B = 120°$, find $\angle BAC$.

3. In $\square ABCD$, $AB = 10$, $\angle B = 30°$ and $AH \perp BC$. Find the length of AH.

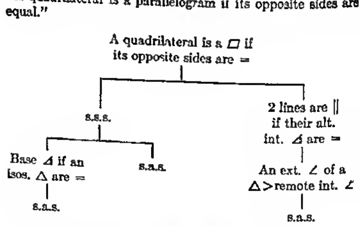
4. In $\square ABCD$, $\angle A = 2\angle B$. Find the number of degrees in $\angle A$.

5. In $\square ABCD$, diagonal $AC \perp BC$ and $AC = BC$. Find the size of $\angle D$.

A proposition's "family tree." By this time you have learned that every proposition in geometry depends on something that precedes it. A later proposition depends directly on those before it which are given as reasons in its proof. These in turn depend in the same way on earlier propositions or on axioms, etc. The whole geometry is built like a chain. Each link supports those which follow it, and a proposition once proved may be used as a reason in proving other propositions.

Let us trace a proposition back by giving first those which are directly used as reasons in its proof, then the propositions used as reasons in proving those, and so on back to the fundamental congruent triangle propositions.

Illustration: Trace the family tree of the proposition: "A quadrilateral is a parallelogram if its opposite sides are equal."



Exercises

Trace back each of the following propositions as above:

1. Two triangles are congruent if the three sides of one . . . .
2. Two lines are parallel if their corresponding angles are equal.
3. Alternate interior angles of parallel lines are equal.

Summary of the Principal Methods

118. Congruent triangles. Two triangles are congruent if:

1. Two sides and the included angle of one equal respectively . . . (§ 50).

2. Two angles and the included side of one equal respectively . . . (§ 52).

3. Three sides of one equal respectively . . . (§ 57).

4 Two angles and a side opposite one of them equal respectively . . . (§ 90).

5. Right triangles, if the hypotenuse and a leg of one equal respectively . . . (§ 94).

119. Isosceles triangles. To prove a triangle isosceles show that:

1. Two sides are equal (§ 40).

2. Two angles are equal (§ 91).

120. Equal lines.

1. Corresponding sides of congruent triangles are equal (§ 22).

2. Sides opposite equal angles in a triangle are equal (§ 91)

3. Opposite sides of a parallelogram are equal (§ 105).

4. A point on the perpendicular bisector of a line is equally distant from the ends (§ 59).

5. A point on the bisector of an angle is equally distant from the sides (§ 97).

6. The diagonals of a parallelogram bisect each other (§ 111).

7. Lines are proved equal by the use of axioms (§ 30).

121. Equal angles. Angles are equal, if they are:

1. Corresponding angles of congruent triangles (§ 22).

2. Right angles (§ 33).

3. Vertical angles (§ 37).

4. Supplements of equal angles (§ 34).

Completion Test *(10 min.)*

Write the numbers of the questions on your answer paper and after each the one word that is omitted.

1. The diagonals of an equilateral parallelogram are . . . to each other.
2. The statement, "A quadrilateral is a parallelogram if its opposite sides are equal," is a . . . .
3. The trapezoid is a special kind of . . . .
4. A line bisecting one side of a triangle and parallel to the second side . . . the third side.
5. If in quadrilateral $ABCD$ $AB=10$, $BC=8$, $DA=8$ and $CD=10$, then the quadrilateral must be a . . . .

True-False Test *(10 min.)*

Copy the numbers of the statements. If the statement is true, write T after its number, if false, write F.

1. The diagonals of a parallelogram divide it into four congruent triangles.
2. The diagonals of a parallelogram cut each other in halves.
3. If two angles of a triangle are equal, the third angle is equal.
4. A quadrilateral is a parallelogram if its diagonals are perpendicular to each other.
5. A quadrilateral is a parallelogram if two sides are equal.
6. A parallelogram, not equilateral, has two unequal altitudes.
7. A trapezoid is isosceles if it has two equal sides.
8. Any point on the bisector of an angle is equally distant from the ends of the angle.
9. If an angle of a right triangle is 30°, the side opposite is half the hypotenuse.
10. An equiangular triangle is equilateral.

appear to be true in these special cases do not hold for the general case.

3. State in terms of your figure what is given and what is to be proved.

4. Decide on a plan of attack.

Do not blindly record every fact that may be true for your figure without considering its use in the proof. First fix clearly in your mind what you must prove. Then ask yourself which of the methods given in sections 118 to 124 seems most likely to give a proof of this.

5. Study the hypothesis and the figure to determine what is known and what is needed to apply this method.

For example, if you must prove triangles congruent, and find one side and one angle in one of them equal to the corresponding parts in the other, pick out the side which will give you two sides and the included angle by § 50 and try to discover a means of proving it equal to its corresponding part. Or try to get another angle so that you can use § 52 or § 90.

6. When you have decided what additional part you need, ask yourself which of the methods for this kind of part seems most likely to give a proof.

For example, if you need another pair of sides equal, ask yourself if there are other triangles of which these same lines are corresponding sides.

7. Work backwards in this way until you arrive at some statement which you can prove.

If, however, you cannot complete a proof with the method chosen, go back to the place where you had a choice of methods and proceed in the same way along another line. But very often each of the methods, which might have been chosen, will result in a proof.

The analytic method of proof amounts to this: Start with the last statement and work backwards. It is not always possible to tell in advance how a proof should begin, but the

5. Base angles of an isosceles triangle (§ 55).

6. Alternate interior or corresponding angles of parallel lines (§§ 79, 80).

7. Third angles of two triangles having two angles of one equal to two angles of the other (§ 89).

8. Angles are proved equal by the use of axioms (§ 30).

122. Parallel lines. Lines are parallel, if:

1. The alternate interior angles are equal (§ 72).
2. The corresponding angles are equal (§ 74).
3. They are parallel or perpendicular to the same line (§§ 82, 75).
4. They are opposite sides of a parallelogram (§ 100).

123. Perpendicular lines. Lines are proved perpendicular by showing that:

1. Two points on one are each equally distant from two points on the other (§ 61).
2. Two supplementary adjacent angles are equal (§§ 11, 12).
3. The sum of two angles of a triangle equals the third angle (§ 86).

124. Parallelograms. A quadrilateral is a parallelogram, if:

1. The opposite sides are parallel (§ 100).
2. The opposite sides are equal (§ 109).
3. Two sides are both equal and parallel (§ 110).
4. The diagonals bisect each other (§ 112).

125. Study guides to the analytic method of discovering a proof.

1. Read the statement carefully, and determine what is given and what is to be proved.

2. Draw an accurate figure representing the facts given.

This figure should be general. For example, when a triangle is given, do not draw an equilateral triangle or an isosceles triangle unless the hypothesis calls for one of these. Many things that

(4) "two angles and a side opposite one of them." Therefore, another angle should be looked for. Try $\angle w = \angle x$. Again, only one method of proving $\angle w = \angle x$ (§ 121) seems to apply, that is: (1) "Corresponding angles of congruent triangles are equal." Write this reason, and then, *as a preceding statement*, $\triangle DEB \cong \triangle FEC$.

It will now be seen that these triangles can be proved congruent by showing that two sides and the included angle of one are equal to two sides and the included angle of the other. So, fill in above this statement the three parts necessary to prove the triangles congruent. Also fill in any reasons previously left blank and the final result is:

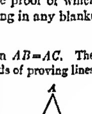
STATEMENTS	REASONS
$\angle DEB = \angle FEC$.	Vert. angles are equal.
$BE = EC$.	Hyp.
$DE = EF$.	Hyp.
$\triangle DEB \cong \triangle FEC$.	Two $\triangle$s are $\cong$ if two sides and the included angle . . .
$\angle w = \angle x$.	Corr. $\angle$s of $\cong$ $\triangle$s are $=$.
$EF = DE$.	Hyp.
$BE = EC$.	*Hyp.*
$BF = DC$.	If equals are added to equals, . . .
$\angle A = \angle A$.	Iden.
$\triangle ABF \cong \triangle ACD$	Two $\triangle$s are $\cong$ if two angles and a side opposite one of them . . .
$AB = AC$.	Corr. sides of $\cong$ $\triangle$s are $=$.

In another form, this proof might be stated as follows: We can prove $AB = AC$, if we can prove $\triangle ABF \cong \triangle ACD$. But, since we have a side and one angle, we can prove $\triangle ABF \cong \triangle ACD$, if we can prove $\angle w = \angle x$. Now we can prove $\angle w = \angle x$, if we can prove $\triangle DEB \cong \triangle FEC$. And we can prove $\triangle DEB \cong \triangle FEC$ by showing that two sides and the included angle of one are equal to two sides and the included angle of the other. Therefore we can prove $AB = AC$.

If we had chosen $\angle y = \angle z$, instead of $\angle w = \angle x$, we should have succeeded by another proof, slightly longer. And, if at the beginning of our work, we had chosen (2) sides opposite equal angles in a

last statement, being the conclusion which was to be proved, is always known. Write this statement at the bottom of the page. Then, the reason for this statement, and consequently the next to the last statement, will generally be one of the methods given on pages 115–116. The nature of the problem will usually limit the number of reasonable lines of attack to one or two, and very often each of these will give a solution. Choose one of them as the next to the last statement, and try to reduce the proof back another statement. If the method chosen first does not give a solution, try another. In this way, you arrive at a statement the proof of which is evident. Then complete the proof by filling in any blanks which were left while working backwards.

Illustration. If $BE=EC$ and $ED=EF$, then $AB=AC$. The last statement is $AB=AC$. Among the methods of proving lines equal (§ 120), an inspection will show that only (1) "corresponding sides of congruent triangles are equal" and (2) "sides opposite equal angles in a triangle are equal" seem reasonable for this problem. Try (1): "Corresponding sides of congruent triangles are equal." Then the next to the last statement must be: $\triangle ABF \cong \triangle ACD$. Now the statements before this one must give the parts necessary to prove the triangles congruent. It will immediately be seen that $\angle A$ is identical, and $DC=BF$, the latter obtained by adding together its parts. The proof now stands:



STATEMENTS	REASONS
$EF=DE$.	Hyp.
$BE=EC$.	Hyp.
$BF=DC$.	Ax. 3.
$\angle A=\angle A$.	Iden.
$\triangle ABF\cong\triangle ACD$.	
$AB=AC$.	Corr. sides of $\cong$ $\triangle$ are equal.

Since an angle and the side opposite have been found, the only reasonable method of proving the triangles congruent (§ 118), is

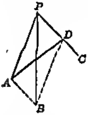
12. AD, BE, and CF are the medians of $\triangle ABC$. If DK is equal and parallel to BE, prove that KA is equal and parallel to CF. *(See* Fig. for Ex. 12 on page 120.)

13. *Three straight lines meet in a point. Draw another straight line so that the parts of it intercepted between the given lines are equal.*

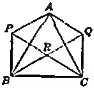
14. On the sides of an equilateral $\triangle ABC$ as bases, congruent *isosceles,* $\triangle ABP$, ACQ, *and* BCR *are constructed. Prove that* $APRQ$ is a rhombus.

15. The base angles of an isosceles trapezoid are equal.

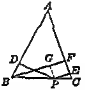
16. A right triangle is cut into two isosceles triangles by the *median to the hypotenuse.*



Ex. 13.



Ex. 14.



Ex. 17.

17. In an isosceles triangle, the sum of the perpendiculars on the legs from any point in the base equals the altitude on one of the legs.

18. *In an equilateral triangle, the sum of the perpendiculars from* any point inside the triangle on the three sides equals the altitude of the triangle.

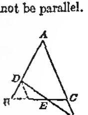
19. The angle formed by the bisector of an angle and the altitude from the same vertex equals one-half the difference of the other two angles of the triangle.

20. The bisectors of the angles of a triangle cannot be parallel.

21. Two quadrilaterals are congruent, if three sides and the two diagonals of one equal respectively the three corresponding sides and the two corresponding diagonals of the other.

22. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.

23. If $AB = AC$ and $BD = CF$, then $DE = EF$.

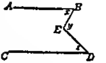


triangle, instead of (1), we should have discovered another proof somewhat easier than the one given. Consequently, in this problem, as in many, it was necessary only to select any reasonable line of attack, and a proof would have followed.

Review Exercises: Honor Work

1. If two quadrilaterals have the four sides and a diagonal of one equal respectively to the four sides and the corresponding diagonal of the other, their other diagonals are equal.

2. The line joining the vertex of an isosceles triangle to the point of intersection of the altitudes on the legs, bisects the vertex angle.



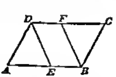
3. If $\angle y = \angle x + \angle z$, prove that $AB \parallel CD$.

4. If $\angle y$ is 1° greater than the sum of $\angle s$ x and z, show which way AB and CD must be extended to meet.

5. The bisectors of the angles of a triangle meet in a point.

6. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

7. If each of the medians BD and CE of $\triangle ABC$ is extended its own length to F and G, respectively, and lines GA and AF are drawn, then GAF is a straight line and A is its middle point.

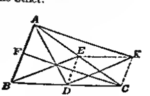


8. If $ABCD$ is a parallelogram, and $DE \parallel FB$, prove that $\triangle AED \cong \triangle BCF$.

9. Angles whose sides are respectively perpendicular are either equal or supplementary.

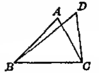
10. Two triangles are congruent, if a side and the altitude and median to that side in one triangle equal respectively a side and the corresponding altitude and median in the other.

11. Two acute triangles are congruent, if two sides and the altitude on the third side in one equal respectively two sides and the corresponding altitude in the other.



Ex. 12.

33. The sides of $\triangle DEF$ are the bisectors of the exterior angles of $\triangle ABC$. Prove that DEF is an acute triangle.



34. If AC equals DC, show that AB cannot equal DB.

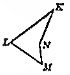
35. Two trapezoids are congruent, if their corresponding sides are equal.

36. If one leg of a trapezoid is perpendicular to the bases, its ends are equally distant from the middle point of the other leg.

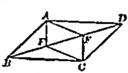
24. In the figure, prove that $\angle KNM = \angle K + \angle L + \angle M$.

25. Two acute triangles are congruent, if a side and the altitudes on the other two sides of one equal respectively a side and the corresponding altitudes of the other.

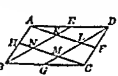
26. In $\square ABCD$, if AE and CF are the bisectors of $\angle DAB$ and BCD respectively, prove that AF equals EC.



Ex. 24.



Ex. 26.



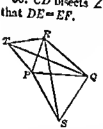
Ex. 27.

27. If E, F, G, and H are the middle points of the sides of $\square ABCD$, prove that $KNML$ is a parallelogram.

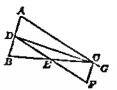
28. PSQ and PRT are equilateral triangles constructed on the sides of any $\triangle PQR$. Prove that $TQ = RS$.

29. ADE and BCF are equilateral triangles on the sides AD and BC of square $ABCD$, outside the square. Prove that $EBFD$ is a parallelogram.

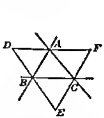
30. CD bisects $\angle ACB$, CF bisects $\angle DCG$, and $DF \parallel AG$. Prove that $DE = EF$.



Ex. 28.



Ex. 30.



Ex. 33.

31. Two lines are parallel, if the bisectors of the interior angles on the same side of a transversal are perpendicular to each other.

32. The middle points of the diagonals of a quadrilateral and the middle points of two opposite sides are the vertices of a parallelogram. (Use § 117.)

133. A secant is a line which cuts the circle in two points; as KN.

134. A tangent is a line which touches a circle at one, and only one, point, however far produced; as QR, which touches at P. The point P is called the *point of contact* or the *point of tangency.*

135. To intercept means to cut off. An angle is said to intercept the arc which its sides cut off.

136. If a circle passes through all the vertices of a polygon, the circle is circumscribed about the polygon, and the polygon is inscribed in the circle.

137. If every side of a polygon is tangent to a circle, the circle is inscribed in the polygon, and the polygon is circumscribed about the circle.

Simple Theorems

138. *Radii of a circle, or of equal circles, are equal.*

139. *Circles are equal, if their radii are equal.*

140. *A point is inside, on, or outside a circle, according as its distance from the center is less than, equal to, or greater than the radius.* (Ax. 8).

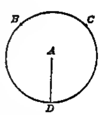
Investigation Problem. In each of two equal circles, draw two radii so that the angles formed at the centers will be equal. Compare the lengths of the arcs. Can you make up a proposition about equal central angles and their arcs? Try to prove your proposition by placing one circle on the other so that the angles coincide. Will the ends of the arcs coincide? Why? Will the circles coincide? Why? Is the converse also true?

In a circle, draw two equal chords. Compare their arcs. How can you prove arcs equal? If you draw radii to the ends of the chords, can you prove the central angles equal?

BOOK TWO

THE CIRCLE

126. A circle is a closed plane curve, all points of which are the same distance from a fixed point in the plane called the center.

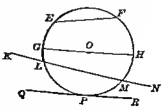


A circle is read by a capital letter placed near its center; as circle (⊙)A. It may also be read by any three of its points; as circle BCD.

127. The circumference is the length of the circle.

128. Any part of the circle is an arc($\frown$); as $\widehat{BC}$.

129. An arc smaller than a semicircle is called a minor arc; one greater than a semicircle is called a major arc. Gener ally, when the word arc is used alone, a minor arc is meant.



130. A radius is a line from the center to any point on the circle; as AD, above.

131. A chord is a line joining two points of the circle; as EF.

132. A diameter is a chord through the center; as GH.

133. A secant is a line which cuts the circle in two points; as KN.

134. A tangent is a line which touches a circle at one, and only one, point, however far produced; as QR, which touches at P. The point P is called the *point of contact* or the *point of tangency*.

135. To intercept means to cut off. An angle is said to intercept the arc which its sides cut off.

136. If a circle passes through all the vertices of a polygon, the circle is circumscribed about the polygon, and the polygon is inscribed in the circle.

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Simple Theorems

138. *Radii of a circle, or of equal circles, are equal.*

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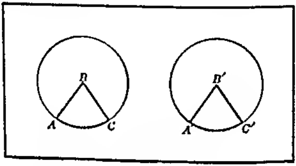
140. *A point is inside, on, or outside a circle, according as its distance from the center is less than, equal to, or greater than the radius.* (Ax. 8).

Investigation Problem. In each of two equal circles, draw two radii so that the angles formed at the centers will be equal. Compare the lengths of the arcs. Can you make up a proposition about equal central angles and their arcs? Try to prove your proposition by placing one circle on the other so that the angles coincide. Will the ends of the arcs coincide? Why? Will the circles coincide? Why? Is the converse also true?

In a circle, draw two equal chords. Compare their arcs. How can you prove arcs equal? If you draw radii to the ends of the chords, can you prove the central angles equal?

Proposition 1

141. *In a circle, or in equal circles, equal central angles have equal arcs.*



Given: $\odot B = \odot B'$, and $\angle B = \angle B'$.
To prove: $\widehat{AC} = \widehat{A'C'}$.

Proof:

Statements	Reasons
1. Place $\odot B$ on $\odot B'$ so that point B is on point B', and BA takes the direction of $B'A'$.	1. Ax. 10.
2. Point A is on point A'.	2. $AB = A'B'$ by § 138.
3. BC takes the direction of $B'C'$.	3. $\angle B = \angle B'$ by hyp.
4. Point C is on point C'.	4. $BC = B'C'$ by § 138.
5. $\widehat{AC}$ coincides with $\widehat{A'C'}$.	5. § 138.
6. $\widehat{AC} = \widehat{A'C'}$.	6. § 21.

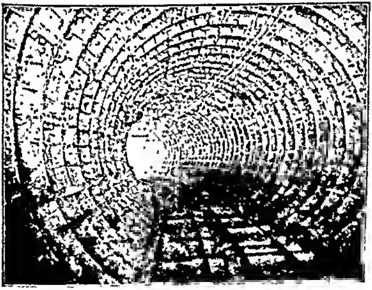
142. Converse. *In a circle, or in equal circles, equal arcs have equal central angles.*

Given: $\odot B = \odot B'$, and $\widehat{AC} = \widehat{A'C'}$.
To prove: $\angle B = \angle B'$.

Proof: STATEMENTS	REASONS
1. Place $\odot B$ on $\odot B'$ so that point B is on point B', and BA takes the direction of $B'A'$.	1. Ax. 10.
2. Point A is on point A'.	2. $AB = A'B'$ by § 138.
3. $\overset{\frown}{AC}$ is on $\overset{\frown}{A'C'}$.	3. $\odot B = \odot B'$ by hyp.
4. Point C is on point C'.	4. $\overset{\frown}{AC} = \overset{\frown}{A'C'}$ by hyp.
5. BC coincides with $B'C'$.	5. § 4.
6. $\angle B = \angle B'$.	6. § 21.

143. Corollary. *A diameter bisects the circle.*

(St. ⦣ are equal, and § 141).



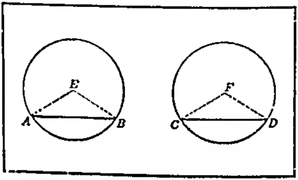
Photograph from Ewing Galloway, N. Y.

THE CIRCLE IN TUNNEL CONSTRUCTION.

This picture shows a section of the subway under the Hudson River at New York City. This subway is constructed from circular rings and is popularly called "The Tube."

Proposition 2

144. *In a circle or in equal circles, equal chords have equal arcs.*



Given: $\odot E = \odot F$, and chord $AB =$ chord CD.
To prove: $\widehat{AB} = \widehat{CD}$.

Proof: Statements	Reasons
1. Draw radii AE, EB, CF, and FD.	1. Post. 1.
2. In $\triangle AEB$ and CFD, chord $AB =$ chord CD.	2. Hyp.
3. $AE = CF$ and $EB = FD$.	3. § 138.
4. $\triangle AEB \cong \triangle CFD$.	4. § 57.
5. $\angle E = \angle F$.	5. § 22.
6 $\widehat{AB} = \widehat{CD}$.	6. § 141.

145. Converse. *In a circle or in equal circles, equal arcs have equal chords.*

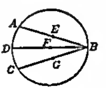
Given: $\odot E = \odot F$ and $\widehat{AB} = \widehat{CD}$.
To prove: chord $AB =$ chord CD.

Proof: STATEMENTS	REASONS
1. Draw radii AE, EB, CF, and FD.	1. Post. 1.
2. In $\triangle AEB$ and CFD, $AE = CF$ and $EB = FD$.	2. § 138.
3. $\widehat{AB} = \widehat{CD}$.	3. Hyp.
4. $\angle E = \angle F$.	4. § 142.
5. $\triangle AEB \cong \triangle CFD$.	5. § 50.
6. Chord AB = chord CD.	6. § 22.

146. Method of attack. *Arcs are proved equal* by means of equal central angles or equal chords. *Lines may be proved equal* by showing that they are chords of equal arcs.

CLASS EXERCISES

1. An inscribed equiangular triangle divides the circle into three equal arcs.



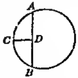
2. If chord AB = chord BC and BD bisects $\angle ABC$, then $\widehat{AD} = \widehat{DC}$.

3. If chord AB = chord BC and $\widehat{AD} = \widehat{DC}$, then BD bisects $\angle ABC$.

4. E, F, and G are the middle points of AB, DB, and BC respectively. If chord AB = chord BC and $EF = FG$, then $\widehat{AD} = \widehat{DC}$.

5. If $\triangle LHK$ is inscribed in a circle and $\angle H = \angle K$, then $\widehat{HL} = \widehat{KL}$.

6. Two equilateral triangles inscribed in a circle are congruent.



7. If CD is the perpendicular bisector of chord AB, then $\widehat{AC} = \widehat{CB}$.

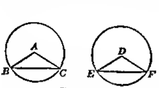
8. If D is the middle point of chord AB and C is the middle point of arc AB, then $CD \perp AB$.

9. If a circle is divided into 360 equal parts, each arc has a central angle of one degree.

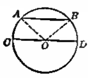
10. If $\odot A = \odot D$ and $\angle B = \angle E$, then $\widehat{BC} = \widehat{EF}$.

11. If four successive chords GH, HI, IJ, and JK are all equal, prove that chord GI equals chord IK.

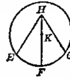
12. In a circle, two inscribed triangles are congruent, if two sides of one equal respectively two sides of the other, and the vertex opposite the longer of these two sides in each triangle is on the major arc cut off by that side.



Ex. 10.



Ex. 18.



Ex. 19.

13. The diagonals of an inscribed parallelogram are equal.

14. The diagonals of an inscribed parallelogram are diameters.

Optional Exercises

15. Prove the converse of Ex. 14.

16. The diagonals of an inscribed equilateral pentagon (five sides) are equal.

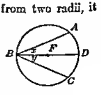
17. If an inscribed polygon is equilateral, it is equiangular.

18. If chord AB is parallel to diameter CD then $\widehat{AC} = \widehat{BD}$.

19. If $HE = HG$ and HF passes through the center K, then $\widehat{EF} = \widehat{FG}$.

20. Isosceles triangles, inscribed in equal circles, are congruent if a leg of one equals a leg of the other.

21. If a point on a circle is equally distant from two radii, it bisects the arc cut off by the radii.



22. If BD is a diameter and $\angle x = \angle y$, then $\widehat{AD} = \widehat{DC}$.

23. If chord AB = chord BC and $\angle x = \angle y$, then BD is a diameter.

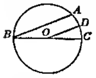
24. If chord AB = chord BC and BD is a diameter, then $\angle x = \angle y$.

25. If two chords bisect each other, the opposite arcs are equal.

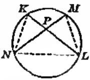
Honor Work

26. Two chords, which bisect each other, are diameters.

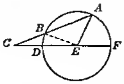
27. If radius OD is parallel to chord BA, then $\widehat{AD} = \widehat{DC}$.



Ex. 27.



Ex. 28.



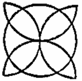
Ex. 29.

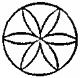
28. If two equal chords intersect, their corresponding parts are equal. (Prove $\triangle KNL \cong \triangle MLN$.)

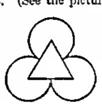
29. If BC equals the radius AE, show that $\angle AEF$ is three times as large as $\angle C$.

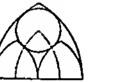
Drawing Exercises

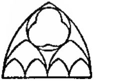
30. Copy the following figures. (See the picture on page 181.)



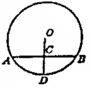






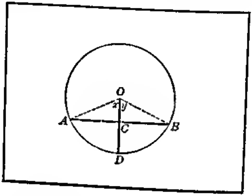


Investigation Problem. In $\odot O$, the radius OD is perpendicular to the chord AB. Compare AC and CB, also arcs AD and DB. What construction lines are needed to prove AC equal to CB? What method have you had of proving arcs equal? Can you prove the central angles equal?



Proposition 3

* **147.** *A radius perpendicular to a chord bisects the chord and its arc.*



Given: [⊙O]; radius $OD \perp$ chord AB.
To prove: $AC = CB$ and $\widehat{AD} = \widehat{DB}$.

Proof:

Statements	Reasons
1. Draw radii OA and OB.	1. Post. 1.
2. In $\triangle AOC$ and BOC, $OA = OB$.	2. § 138.
3. $OC = OC$.	3. Iden.
4. $\angle OCA$ and $\angle OCB$ are rt. angles.	4. $OD \perp AB$ by hyp.
5. $\triangle AOC \cong \triangle BOC$.	5. § 94.
6. $AC = CB$ and $\angle x = \angle y$.	6. § 22.
7. $\widehat{AD} = \widehat{DB}$.	7. § 141.

* **148. Corollary.** *A diameter perpendicular to a chord bisects the chord and both arcs.*

Does a line from the center perpendicular to a chord and ending in the chord necessarily bisect the chord? Prove your answer.

Class Exercises

1. *A diameter bisecting a chord which is not a diameter is perpendicular to the chord.*

2. A diameter bisecting an arc is the perpendicular bisector of the chord of that arc.

3. The perpendicular bisector of a chord passes through the center of the circle.

4. A line bisecting both arcs, into which a chord divides the circle, is the perpendicular bisector of the chord.

5. A line bisecting a chord and its arc passes through the center of the circle.

6. *Bisect a given arc.*

7. *Construct the center of a given circle.*

Optional Exercises

8. In a triangle, no side of which is a diameter of its circumscribed circle, a median through the center of that circle is an altitude.

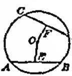
9. The altitude to the base of an isosceles triangle passes through the center of its circumscribed circle.

10. The arcs intercepted between parallel chords are equal.

Honor Work

11. If a chord of a circle moves but remains parallel to a given line, what line will its middle point trace? Prove your statement.

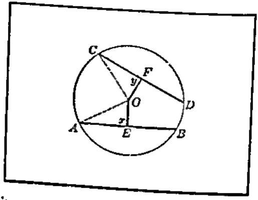
12. If radii OE, OF, OG and OH are drawn perpendicular respectively to the sides AB, BC, CD and DA of a square $ABCD$ inscribed in circle O, and the points E, F, G and H are joined in order, then $EFGH$ is a square.



Investigation Problem. In this figure, AB and CD are equal chords, and OE and OF are the perpendiculars on them from the center. Are OE and OF equal? If you think so, what lines would you draw to prove it? How does AE compare in length with AB? Why? CF with CD? AE with CF? Why? Is the converse of this true? Prove your answer.

Proposition 4

* **149.** *In a circle or in equal circles, equal chords are equally distant from the center.*



Given: $\odot O$, chord $AB =$ chord CD, $OE \perp AB$, and $OF \perp CD$.

To prove: $OE = OF$.

Proof:

Statements	Reasons
1. Draw radii OA and OC.	1. Post. 1.
2. In $\triangle OAE$ and OCF, $OA = OC$.	2. § 138.
3. $\angle x$ and $\angle y$ are rt. angles	3. § 12.
4. $AB = CD$.	4. Hyp.
5. $AE = \frac{1}{2}AB$ and $CF = \frac{1}{2}CD$.	5. § 147.
6. $AE = CF$.	6. Ax. 6.
7. $\triangle OAE \cong \triangle OCF$.	7. § 94.
8. $OE = OF$.	8. § 22.

* **150. Converse.** *In a circle or in equal circles, chords equally distant from the center are equal.*

Given: $\odot O$, $OE = OF$, $OE \perp AB$, and $OF \perp CD$.

To prove: $AB = CD$.

Proof: STATEMENTS	REASONS
1. Draw radii OA and OC.	1. Post. 1.
2. In $\triangle OAE$ and OCF, $OA = OC$.	1. § 138.
3. $\angle x$ and $\angle y$ are rt. angles.	3. § 12.
4. $OE = OF$.	4. Hyp.
5. $\triangle OAE \cong \triangle OCF$.	5. § 94.
6. $AE = CF$.	6. § 22.
7. $AE = \frac{1}{2}AB$ and $CF = \frac{1}{2}CD$.	7. § 147.
8. $AB = CD$.	8. Ax. 5.

151. Method of attack. *Chords are proved equal* by showing that they have equal arcs, or that they are equally distant from the center.

CLASS EXERCISES

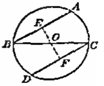
1. A polygon, inscribed in a circle, is equilateral, if the perpendiculars from the center to its sides are equal.

2. If a parallelogram is inscribed in a circle, the opposite sides are equally distant from the center.

3. If perpendiculars from the center to two chords are equal, the arcs of the chords are equal.

4. If two chords from a point on a circle make equal angles with the radius to the point, the chords are equal.

5. The perpendicular bisectors of the sides of an inscribed polygon meet in a point.



6. Two parallel chords, through the ends of a diameter, are equal.

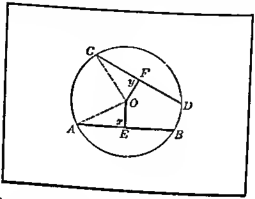
7. Two chords, perpendicular to a third chord at its ends, are equal.

8. $ABCD$ is an inscribed square, and OE, OF, OG, and OH are the radii perpendicular to AB, BC, CD, and DA respectively. Prove that polygon $AEBFCGDH$ is equilateral.

9. In a circle, chords unequally distant from the center are unequal, and conversely.

Proposition 4

* **149.** *In a circle or in equal circles, equal chords are equally distant from the center.*



Given: $\odot O$, chord $AB =$ chord CD, $OE \perp AB$, and $OF \perp CD$.

To prove: $OE = OF$.

Proof:

Statements	Reasons
1. Draw radii OA and OC.	1. Post. 1.
2. In $\triangle OAE$ and OCF, $OA = OC$.	2. § 138.
3. $\angle x$ and $\angle y$ are rt. angles	3. § 12.
4. $AB = CD$.	4. Hyp.
5. $AE = \frac{1}{2}AB$ and $CF = \frac{1}{2}CD$.	5. § 147.
6. $AE = CF$.	6. Ax. 6.
7. $\triangle OAE \cong \triangle OCF$.	7. § 94.
8. $OE = OF$.	8. § 22.

* **150. Converse.** *In a circle or in equal circles, chords equally distant from the center are equal.*

Given: $\odot O$, $OE = OF$, $OE \perp AB$, and $OF \perp CD$.

To prove: $AB = CD$.

Proof: STATEMENTS	REASONS
1. Draw radii OA and OC.	1. Post. 1.
2. In $\triangle OAE$ and OCF, $OA = OC$.	1. § 138.
3. $\angle x$ and $\angle y$ are rt. angles.	3. § 12.
4. $OE = OF$.	4. Hyp.
5. $\triangle OAE \cong \triangle OCF$.	5. § 94.
6. $AE = CF$.	6. § 22.
7. $AE = \frac{1}{2}AB$ and $CF = \frac{1}{2}CD$.	7. § 147.
8. $AB = CD$.	8. Ax. 5.

151. Method of attack. *Chords are proved equal* by showing that they have equal arcs, or that they are equally distant from the center.

CLASS EXERCISES

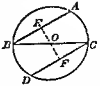
1. A polygon, inscribed in a circle, is equilateral, if the perpendiculars from the center to its sides are equal.

2. If a parallelogram is inscribed in a circle, the opposite sides are equally distant from the center.

3. If perpendiculars from the center to two chords are equal, the arcs of the chords are equal.

4. If two chords from a point on a circle make equal angles with the radius to the point, the chords are equal.

5. The perpendicular bisectors of the sides of an inscribed polygon meet in a point.



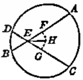
6. Two parallel chords, through the ends of a diameter, are equal.

7. Two chords, perpendicular to a third chord at its ends, are equal.

8. $ABCD$ is an inscribed square, and OE, OF, OG, and OH are the radii perpendicular to AB, BC, CD, and DA respectively. Prove that polygon $AEBFCGDH$ is equilateral.

9. In a circle, chords unequally distant from the center are unequal, and conversely.

10. If the radii perpendicular to two chords of a circle form equal angles with the radii to an end of each chord, the chords are equal.



11. If two intersecting chords make equal angles with the radius to the point of intersection, the chords are equal.

12. State and prove the converse of Ex. 11.

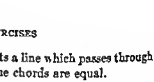
Optional Exercises

13. If the center of the circle bisects a line which passes through it and ends in two parallel chords, the chords are equal.

14. If two equal chords are parallel, the center of the circle bisects any line through it and ending in the chords.

15. Two circles are equal, if their chords equally distant from the centers are equal.

16. If a secant cuts two circles which have the same center, its parts, intercepted between the two circles, are equal.

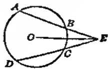


17. If, through two equal circles, a straight line is drawn parallel to the line joining their centers, the chords intercepted in the two circles are equal.

18. In a given circle, construct a chord equal and parallel to a given chord.

19. In a given circle, construct a chord equal to a given chord and parallel to another given chord.

20. Given an arc, find the center of the circle of which it is an arc.

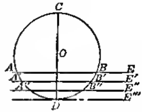


Honor Work

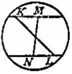
21. If ABE and DCE make equal angles with OE, in $\odot O$, then $AB=CD$.

22. If two equal chords are extended to meet outside the circle, the line joining their point of intersection to the center bisects the angle which they form.

23. If any number of parallel chords are drawn in a circle, their middle points are all in the same straight line.



24. If the secant AE moves away from the center of $\odot O$, but remains always perpendicular to the diameter CD, how do the points A and B move? Compare the distances AD and DB. What can you say about the secant AE when it reaches the end D of diameter CD? What has become of points A and B? What then is the relation of a diameter to the tangent at its end?



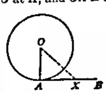
25. Two lines, KL and MN are bisected by the center of a circle. Prove that the chord through K and M equals the chord through N and L.

Applied Problems

26. A wheelwright is given a part of a broken wheel, and told to make a new wheel of the same size. Show how he could construct the center and radius.

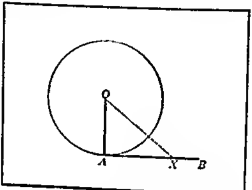
27. A log is sawed so as to yield eight boards and two slabs, each 1 in. thick, one cut passing through the center of the log. Show that the boards can be divided into five pairs, the two boards of each pair having the same width.

Investigation Problem. AB is tangent to $\odot O$ at A, and OA is a radius. What do you suspect about $\angle OAB$? Is $OA \perp AB$? What do you know about the length of a perpendicular to a line as compared with the length of any other line from the same point to that line? Is OA shorter than any other line from O to AB? Why? Is X necessarily outside the circle?



Proposition 5

152. *A tangent is perpendicular to the radius drawn to the point of contact.*



Given: [$\odot O$]; tangent AB and radius OA to the point of contact A.

To prove: $AB \perp OA$.

Proof:

Statements	Reasons
1. Draw OX to any point X of AB except A.	1. Post. 1.
2. X is outside $\odot O$.	2. § 134.
3. $OA < OX$.	3. § 140.
4. $OA \perp AB$ or $AB \perp OA$.	4. Ax. 12.

Investigation Problem. State the converse of Proposition 5. Do you think it is true? In the above figure, to prove AB a tangent, what must you prove about any point on it? How does the length of OX compare with that of OA? Why? Where then is point X? Can you draw more than one line tangent to a circle through a point on the circle? Can you draw more than one line tangent to a circle through a point outside the circle? Assuming that there is a tangent to a circle at a point on the circle, prove indirectly that a perpendicular to a radius is that tangent.

153. Converse. *A line perpendicular to a radius at its outer end is tangent to the circle.*

Given: [$\odot O$]; radius OA and $AB \perp OA$.

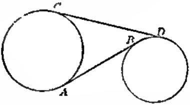
To prove: AB is tangent to $\odot O$.

Proof: STATEMENTS	REASONS
1. Draw OX to any point X of AB except A.	1. Post. 1.
2. $OX > OA$.	2. Ax. 12.
3. X is outside $\odot O$.	3. § 140.
4. AB is tangent to $\odot O$.	4. § 134.

154. Corollary 1. *A perpendicular to a tangent at the point of contact passes through the center of the circle.* (Use § 152.)

If a line is drawn from an outside point to the center of a circle, the length of the segment from the point to the nearest intersection with the circle is called the distance of that point from the circle.

155. Two circles are tangent to each other, if both are tangent to the same line at the same point.



A common tangent to two circles is internal if it passes between the circles, as AB; and external if it does not pass between the circles, as CD.

156. Corollary 2. *If two circles are tangent to each other, the point of contact and the two centers lie in a straight line.* (Use § 154.)

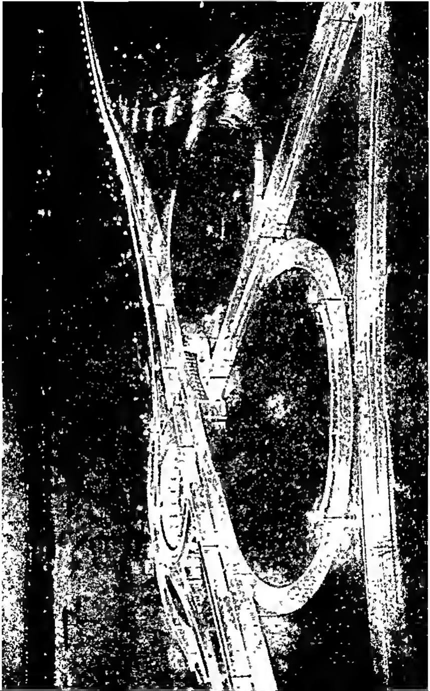
Investigation Problem. Draw two tangents to a circle from an outside point. How do they compare in length, measured from the outside point to the point of contact? Can you draw two tangents to a circle from an outside point such that their lengths are not equal? State and prove your conclusion.

GEOMETRY IN MODERN ROAD BUILDING

Because of the speed at which cars travel today, the problem of road building becomes more and more scientific, and consequently a better knowledge of geometry is required of the engineer who plans the project. In the picture shown here, the circular roads tangent to the straight-line speedways are needed to conduct traffic from either roadway in either direction to the other road in either direction without making it necessary for any line of traffic to cross any other. This is one of the problems of modern road building.

Another problem of road construction that requires a knowledge of geometry is that of banking a curve. The amount that the road must be raised on the outside of the curve depends on the sharpness of the curve, that is, on the length of the radius of the circle. You will learn in Book 3 how this radius is determined. At the present time many roads throughout the country are not banked the right amount because the man in charge of the construction did not have the proper scientific training. So there is increasing need for persons with a good geometric background.

Photo by Paul J. Woolf



GEOMETRY IN MODERN ROAD BUILDING

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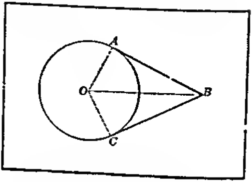
Photo by Paul J. Woolf

Proposition 6

157. *Tangents to a circle from a point are equal.*



Given: AB and BC tangent to $\odot O$.
To prove: $AB = BC$.

Proof:

Statements	Reasons
1. Draw OA, OB, and OC.	1. Post. 1.
2. In $\triangle ABO$ and CBO, $BO = BO$.	2. Iden.
3. $OA = OC$.	3. § 138.
4. $OA \perp AB$ and $OC \perp BC$.	4. § 152.
5. $\triangle ABO \cong \triangle CBO$.	5. § 94.
6. $AB = BC$.	6. § 22.

Class Exercises

1. Tangents to a circle from a point form equal angles with the chord joining their points of contact.

2. If two tangents are drawn to a circle from a point, the line joining that point to the center of the circle bisects the angle made by the tangents.

3. Tangents [illegible] the ends of a diameter are parallel.

4. Draw two circles, so that the distance between their centers shall be:

(a) greater than the sum of the radii.

(b) equal to the sum of the radii.

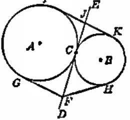
(c) less than the sum of the radii, but greater than their difference.

(d) equal to the difference of the radii.

(e) less than the difference of the radii, but greater than zero.

(f) equal to zero.

5. How many common tangents can be drawn to the two circles in each of the above cases?



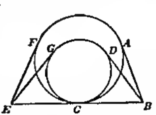
6. If ⊙A and B are tangent to DE at C, show that tangents FG and FH, from any point F of DE, are equal.

7. If two circles are externally tangent to each other, their common internal tangent bisects their common external tangents.

8. The common internal tangents to two circles are equal.

9. The common external tangents to two unequal circles are equal.

10. If AB and CD are the common internal tangents to two circles, then chord AC is parallel to chord DB.



11. AB, BE, BD, EG, and EF are all tangents, and the circles are tangent to each other at C. Then $BD = EG$, if $AB = EF$.

12. A chord is parallel to a tangent drawn at the middle point of its arc.

13. Construct a tangent parallel to a given chord.

14. If two tangents meet at an angle of 60°, the chord joining their points of contact equals each tangent.

Proposition 6

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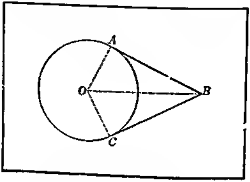

Given: AB and BC tangent to $\odot O$.
To prove: $AB = BC$.

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Statements	Reasons
1. Draw OA, OB, and OC.	1. Post. 1.
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Class Exercises

1. Tangents to a circle from a point form equal angles with the chord joining their points of contact.

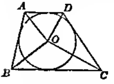
2. If two tangents are drawn to a circle from a point, the line joining that point to the center of the circle bisects the angle made by the tangents.

3. Tangents to a circle at the ends of a diameter are parallel.

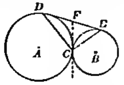
Honor Work

26. $ABCD$ is a circumscribed quadrilateral. If $AD \parallel BC$, prove that $\measuredangle AOB$ and DOC are right angles.

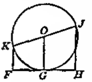
27. The radius of the circle inscribed in an equilateral triangle is one-third of the altitude. (Use § 95.)



Ex. 26.



Ex. 28.



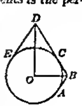
Ex. 29.

28. $\odot A$ and B are tangent at C, and DE is a common tangent. Prove that $\angle DCE$ is a right angle.

29. *If FH is tangent to $\odot O$ at G, KJ any diameter, $KF \perp FH$, and $JH \perp FH$,* prove that $FG = GH$.

30. The angle formed by two tangents is the supplement of the angle formed by the radii to the points of contact.

31. *The bisector of the angle formed by two tangents is the perpendicular bisector of the chord joining their points of contact.*



32. The bisectors of the angles of a circumscribed polygon are the perpendicular bisectors of the sides of the inscribed polygon, whose vertices are the points of contact of the sides of the circumscribed polygon.

33. If AB, BD, and DE are tangents to $\odot O$, and $OB \perp OD$, then $AB \parallel DE$.

Applied Problems

34. Two streets meet at an angle of 45°. It is proposed to round off the point of the curb by constructing an arc of a circle of radius ten feet tangent to both curbs. Construct a plan for the work.

15. $\triangle ABC$ is a circumscribed triangle, D being the point of contact of BC. If $BD = DC$, then $\angle B = \angle C$.

Optional Exercises

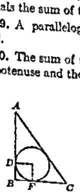
16. A triangle is equilateral, if its inscribed and circumscribed circles have the same center.

17. Two tangents are drawn to a circle from a point outside. Prove that the triangle formed by these tangents and any tangent to the arc included by them, has a perimeter equal to the sum of the first two tangents.

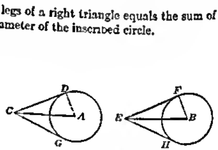
18. The sum of two opposite sides of a circumscribed quadrilateral equals the sum of the other two sides.

19. A parallelogram circumscribed about a circle is equilateral.

20. The sum of the legs of a right triangle equals the sum of the hypotenuse and the diameter of the inscribed circle.



Ex. 20.



Ex. 23.

21. A circumscribed rectangle is a square.

22. The perpendiculars to the sides of a circumscribed polygon at their points of contact pass through a common point.

23. If tangent CD equals tangent EF and $\angle DCG = \angle FEH$, then $\odot A = \odot B$.

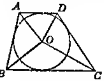
24. The bisector of the angle formed by two tangents to a circle passes through the center of the circle.

25. The bisectors of the angles of a circumscribed quadrilateral meet in a common point.

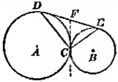
Honor Work

26. $ABCD$ is a circumscribed quadrilateral. If $AD \parallel BC$, prove that $\angle s$ AOB and DOC are right angles.

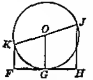
27. The radius of the circle inscribed in an equilateral triangle is one-third of the altitude. (Use § 95.)



Ex. 26.



Ex. 28.



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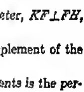


32. The bisectors of the angles of a circumscribed polygon are the perpendicular bisectors of the sides of the inscribed polygon, whose vertices are the points of contact of the sides of the circumscribed polygon.

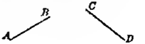
33. If AB, BD, and DE are tangents to $\odot O$, and $OB \perp OD$, then $AB \parallel DE$.

Applied Problems

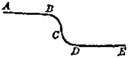
34. Two streets meet at an angle of 45°. It is proposed to round off the point of the curb by constructing an arc of a circle of radius ten feet tangent to both curbs. Construct a plan for the work.

35. A running track has two straight parallel paths, connected by two half circles, tangent to both. Construct a plan of the track.

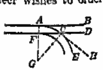
36. In building a railroad, two straight sections of track, AB and CD, are to be connected by the arc of a circle, tangent to CD, and tangent to AB at B. Construct the center of the circle.



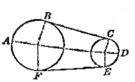
37. A railroad engineer must construct a track passing through C, and joining the two parallel tracks AB and DE, so he plans to have the arc of a circle tangent to AB at B, and passing through point C, then forming a reverse curve, which is another circular arc, tangent to the arc BC at C and tangent to the straight line DE. Construct the plan for this track.



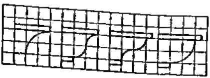
38. In building a railroad switch, the engineer wishes to order a frog, which is the piece of track composed of the rails ACE and FCD near the point C where they cross. He must state the number of degrees in $\angle DCH$ which is formed by the straight rail and the tangent CH to the curved rail ACE. He has already determined $\angle AGC$ made by the radii AG and CG. Prove that $\angle DCH$ equals $\angle AGC$.



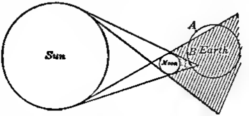
39. A mechanic wishes to find the length of belt for two pulleys, so he measures the length $ABCD$. Show that the length of the belt will be exactly twice this.



40. Construct the following moldings. The arcs are quarter circles having their centers at the vertices of squares.



41. During an eclipse of the sun, the sun, moon, and earth are in the positions shown in the diagram. In determining facts about the eclipse, the astronomer makes use of the four common tangents to the sun and moon. There is no eclipse at A, a partial eclipse at B, and a total eclipse at C. Draw a diagram showing how the sun would appear if you were at B. Draw a figure illustrating an eclipse of the moon, that is, when the earth is between the sun and the moon.



A Self-Measuring Test

1. Give four methods that depend on circles for proving lines equal.

2. Give two methods that depend on circles for proving angles equal.

3. Define: *tangent; chord; secant; arc; circle; radius.*

4. Make a list of ten methods of proving angles equal.

5. State two propositions about tangents.

6. Define *parallel lines.*

7. Give three methods of proving lines parallel.

8. Define: (*a*) *supplementary angles;* (*b*) *vertical angles;* (*c*) *straight angle.*

9. Make a list of eight methods of proving lines equal.

10. Define *isosceles triangle.*

11. Give two propositions about isosceles triangles.

12. State two propositions in which chords are proved equal.

13. State two propositions in which arcs are proved equal.

14. If a chord and a tangent have a point in common, where is that point?

15. Give four methods of proving triangles congruent.

16. What are the main purposes of proving triangles congruent?

17. Define a *parallelogram*.

18. Give four facts about parallelograms which you have proved

19. Give two methods of proving one line perpendicular to another.

20. Are congruent figures necessarily equal? Are equal figures necessarily congruent?

21. Why do we say that circles are equal instead of congruent? Are equal circles necessarily congruent?

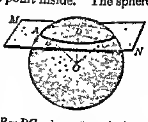
22. In what way can the definition of the circle be used to prove lines equal?

23. What method of proving lines equal depends on tangents?

Space Geometry (*Optional*)

The sphere. The sphere is a closed surface all points of which are the same distance from a point inside. The sphere is like a rubber ball.

All radii of a sphere are equal.



Exercises

1. Prove that a section of a sphere made by a plane is a circle.

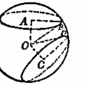
Draw OD from the center of the sphere $\perp$ plane MN. Then prove $DB = DC$ where B and C are any two points on the curve.

2. If plane MN moves ... of the sphere, how does the size of $\odot D$ change? ... the largest? How does its radius then compare ...

3. Since there are no ... between two points must be ... line small circle or ... of a two points? Why ... circle through ... board. ...

4. The section of a sphere made by a plane which passes through the center of the sphere is called a great circle. How many great circles can pass through two points on a sphere, (*a*) if the points are ends of a diameter; (*b*) if not?

5. An aviator, flying from Chicago to Rome, Italy, was forced to the land near Hudson Bay. Why did he go away up there instead of staying near the latitude of Chicago or Rome?



6. If two circles of a sphere have their planes equally distant from the center of the sphere, are their radii necessarily equal? Prove your answer.

7. If a plane is perpendicular to a radius of a sphere at its outer end, what is its relation to the sphere?

8. Make up propositions for the sphere resembling those of paragraphs 152, 153, 154 and 156. Do they appear to be true?

9. Can more than one plane be drawn tangent to a sphere at a given point, (*a*) on the sphere; (*b*) not on the sphere?

10. Can more than one line be drawn tangent to a sphere at a given point, (*a*) on the sphere; (*b*) not on the sphere?

Completion Test (*10 min.*)

Copy the numbers of the questions and after each write the one word that is omitted.

1. If two circles touch each other externally, the greatest number of common tangents that can be drawn is . . . .

2. Tangents to a circle at the ends of a diameter are . . .

3. A diameter, bisecting a chord which is not a diameter, is . . . to the chord.

4. If two unequal circles have the same center, all chords of the larger circle that are tangent to the smaller circle are . . . .

5. A line through the center of a circle perpendicular to a chord . . . the chord.

6. Two chords of a circle equally distant from the center are . . . .

7. A point is inside a circle if its distance from the center is . . . than the radius.

8. If two circles are tangent to each other, the point of contact and the two centers lie on a . . . .

16. What are the main purposes of proving triangles congruent?

17. Define a *parallelogram*.

18. Give four facts about parallelograms which you have proved

19. Give two methods of proving one line perpendicular to another.

20. Are congruent figures necessarily equal? Are equal figures necessarily congruent?

21. Why do we say that circles are equal instead of congruent? Are equal circles necessarily congruent?

22. In what way can the definition of the circle be used to prove lines equal?

23. What method of proving lines equal depends on tangents?

SPACE GEOMETRY (*Optional*)

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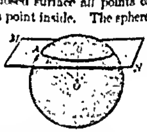
All radii of a sphere are equal.



EXERCISES

1. Prove that a section of a sphere made by a plane is a circle.

Draw OD from the center of the sphere $\perp$ plane MN. Then prove $DB = DC$ where B and C are any two points on the curve.

2. If plane MN moves toward the center of the sphere, how does the size of $\odot D$ change? When is $\odot D$ the largest? How does its radius then compare with the radius of the sphere?

3. Since there are no straight lines on a sphere, the shortest line between two points must be a curve. Do you think the arc of a small circle or of a very large circle would be the shorter line between two points? Why? Test by drawing a small circle and a large circle through the same two points either on paper or on the blackboard. Which arc is more nearly straight? Which is shorter?

True-False Test (10 min.)

Write T if the statement is true, F if it is false.

1. A tangent is a line that touches a circle at one and only one point, however far produced.
2. In a circle central angles have equal arcs.
3. A line perpendicular to a radius is tangent to the circle.
4. A radius perpendicular to a chord bisects the chord.
5. In a circle equal chords have equal arcs.
6. Equal chords are equally distant from the centers of circles.
7. Tangents to a circle are equal.
8. A tangent is perpendicular to the radius to the point of contact.
9. Any part of a circle is an arc.
10. Radii of circles are equal.

158. ***An inscribed angle*** *is an angle formed by two chords meeting on the circle.*

159. **A segment of a circle is the figure formed by an arc and its chord. An angle is inscribed in a segment, if its vertex is on the arc of the segment and its sides are chords drawn to the ends of that arc.**

$\angle B$ is inscribed in segment in ABC.

160. Measurement of arcs. **The arc of a circle intercepted by a central angle of one degree is called** ***an arc degree.***

The arc degree has the same subdivisions as the angle degree; that is, $60'' = 1'$, $60' = 1°$. An arc of 90° is a quadrant.

161. An angle is **measured by** an arc, when the number of angle degrees in the angle equals the number of arc degrees in the arc.

For example, in the above figure, if $\angle A$ contains 42 angle degrees and $\widehat{BC}$ contains 84 arc degrees, we say that $\angle A$ is measured by half of $\widehat{BC}$, which means simply that the number 42 is half of the number

MATCHING TEST (*10 min.*)

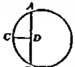
Copy the numbers 1 to 10. After each write the letter of the phrase that is the correct definition of the word following the number.

1. Minor arc	*a.* A part of a circle.
2. Secant	*b.* The length of the circle.
3. Inscribed circle	*c.* A line touching a circle at one point.
4. Circumference	*d.* A line joining two points on a circle.
5. Arc	*e.* A circle passing through all vertices of a polygon.
6. Tangent	*f.* A chord through the center of a circle.
7. Diameter	*g.* To cut off.
8. Chord	*h.* An arc less than a semicircle.
9. Circumscribed circle	*i.* A line intersecting a circle.
10. Intercept	*j.* A circle touching all sides of a polygon.

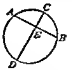
JUDGING THE CORRECTNESS OF A CONCLUSION (*10 min.*)

If the conclusion is correct, give the part of the hypothesis used in proving it; if false, write F.

Exs. 1, 2



Exs. 3, 4



Ex. 5

1. Given: $OX > OA$ and $AX = XB$. Then AB is tangent to $\odot O$.

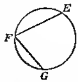
2. Given: AB tangent to $\odot O$ at A, $OX < AB$, O the center of $\odot O$, and $OA = AX$. Then $OA \perp AB$.

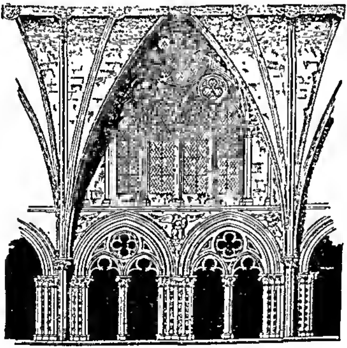
3. Given: $\widehat{AC} = \widehat{CB}$ and $CD \perp AB$. Then $AD = DB$.

4. Given: $CD = \frac{1}{2}AB$ and $CD \perp AB$. Then $\widehat{AC} = \widehat{CB}$.

5. Given: $\widehat{AC} = \widehat{CB}$ and $AE = EB$. Then $\widehat{AD} = \widehat{DB}$.

cannot tell directly, what do you know about the relative sizes of $\angle A$ and $\angle B$? How does $\angle AOC$ compare with $\angle A$ and $\angle B$? What then is $\angle B$ measured by? In the second figure, neither side of the angle is a diameter. Can you draw a line dividing the angle into two parts, each of which will have one side a diameter? What is each part measured by? What then is the whole angle measured by? If both chords EF and FG were on the same side of the center of the circle, would your proof still hold? Are you a good enough student to complete this proof before looking at Proposition 7?



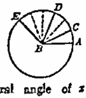


THE USE OF THE CIRCLE IN DESIGN.

This picture is of part of the interior of the cathedral at Lincoln, England.

84. Therefore, since *is measured by* means that one number is equal to another, the axioms and propositions about equals may be used as reasons.

Investigation Problem. If a central angle of 1° intercepts an arc of 1°, a central angle of 2° will intercept an arc of........, for if it were cut into two equal angles of......, each would intercept an arc of........ A central angle of 3° will intercept an arc of.. .., because if it were cut into...... equal parts, each would intercept an arc of. . .. A central angle of 10° will intercept an arc of......... Similarly a central angle of x degrees will intercept an arc of........



A central angle of $\frac{1}{2}°$ will intercept an arc of........, for if a central angle of 1° were cut into........equal parts, the arc of 1° would be cut into........equal arcs. Similarly, a central angle of $\frac{1°}{n}$ will intercept an arc of........, for if a central angle of 1° were cut into........equal parts, the intercepted arc of 1° would be cut into........equal arcs.

We shall assume, then, as an axiom that:

162. *A central angle is measured by its intercepted arc.*

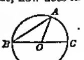
163. Corollary 1. *A circle contains 360°.*

164. Corollary 2. *A right angle is measured by half a semicircle.*

Exercises

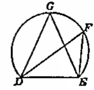
1. As an arc grows larger, what change takes place in its central angle? When the arc becomes twice its original size, how does its central angle compare with its original size?

2. Does the same relation hold for an arc and its chord?



Investigation Problem. By what arc is central $\angle AOC$ measured? How does $\angle B$ compare with $\angle AOC$? If you

167. Corollary 2. *Angles inscribed in the same segment are equal.*



168. Corollary 3. *The opposite angles of an inscribed quadrilateral are supplementary.*

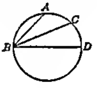
169. Method of attack. *Angles may be proved equal* by showing that they are measured by equal arcs.

Class Exercises

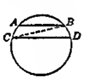
1. Is an angle degree always the same size? Is an arc degree always the same size?

2. If BD is a diameter and $\widehat{AB}=80°$, find $\angle ABD$.

3. If $\angle ABD=45°$ and $\widehat{CD}=40°$, find $\angle ABC$.



Exs. 2, 3.



Ex. 8.

4. If chords AB and CD intersect at E, show that $\triangle AEC$ and DEB are mutually equiangular.

5. The sides of a triangle inscribed in a circle cut off arcs which have the ratio $2:3:4$. Find the number of degrees in each angle of the triangle.

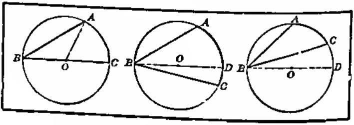
6. An angle inscribed in a segment less than a semicircle is obtuse, and one inscribed in a segment greater than a semicircle is acute.

7. In a circle or in equal circles, inscribed acute triangles are congruent, if two sides of one equal respectively two sides of the other.

8. If four points on a circle are in the order C, A, B, D and if $\widehat{AC}=\widehat{BD}$, prove that chord $AB \parallel$ chord CD.

Proposition 7

* 166. *An inscribed angle is measured by half its intercepted arc.*



Given: $\angle ABC$ inscribed in $\odot O$.

To prove: $\angle ABC$ is measured by $\frac{1}{2}\widehat{AC}$.

Proof:

Statements	Reasons
Case 1. *When one side is a diameter.*	
1. Draw OA.	1. Post. 1.
2. $OA = OB$.	2. § 138.
3. $\angle A = \angle B$.	3. § 55.
4. $\angle B + \angle A = \angle AOC$.	4. § 87.
5. $2\angle B = \angle AOC$.	5. Ax. 1.
6. $\angle AOC$ is measured by $\widehat{AC}$.	6. § 162.
7. $\angle B$ is measured by $\frac{1}{2}\widehat{AC}$.	7. Ax. 6.
Case 2. *When neither side is a diameter.*	
1. Draw the diameter BD.	1. Post. 1.
2. $\angle ABD$ is measured by $\frac{1}{2}\widehat{AD}$ and $\angle DBC$ is measured by $\frac{1}{2}\widehat{DC}$.	2. Case 1.
3. $\angle ABC$ is measured by $\frac{1}{2}\widehat{AC}$.	3. Ax. 3 or 4.

166. Corollary 1. *An angle inscribed in a semicircle is a right angle.* ($\angle B$ is measured by $\frac{1}{2}$ a semicircle, § 165.)

OPTIONAL EXERCISES

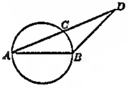
20. If, from a point A on a circle, a chord AB and a diameter AC are drawn, a diameter parallel to AB bisects arc BC.

21. If two secants from a point to a circle are equal, their chord parts are equal.

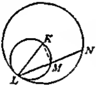
22. An inscribed parallelogram is a rectangle.

23. If a circle is cut into two segments, the angle inscribed in one segment is the supplement of the angle inscribed in the other.

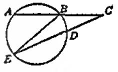
24. If AB is the diameter, and AC equals CD, prove that BD equals AB.



Ex. 24.



Ex. 28.



Exs. 30, 31.

25. The line, drawn from the vertex of the right angle to the middle point of the hypotenuse of a right triangle, equals one-half the hypotenuse. (Circumscribe a circle.)

26. A circle, described on one of the equal sides of an isosceles triangle as a diameter, bisects the base.

27. A circle, drawn on one side of an equilateral triangle as a diameter, bisects the other two sides of the triangle.

28. The diameter of one circle is the radius of a second circle. Prove that all chords of the larger circle, drawn from the point of tangency of the two circles, are bisected by the smaller circle.

29. If a circle is divided into four equal parts, the chords joining the points of division in succession form a square.

30. If $\widehat{AE} = 100°$ and $\widehat{BD} = 30°$, find the number of degrees in $\angle C$.

31. The angle formed by two secants meeting outside a circle is measured by one-half the difference of the intercepted arcs.

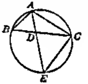
9. Prove the converse of Exercise 8.

10. If AE bisects $\angle A$ of $\triangle ABC$, prove that $\triangle ABD$ and AEC are mutually equiangular.

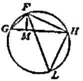
11. Two acute triangles inscribed in a circle are congruent, if a side and one adjacent angle of one equal respectively a side and the corresponding angle of the other.

12. Two triangles inscribed in a circle are congruent, if two angles of one equal two angles of the other.

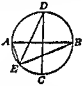
13. Two isosceles triangles inscribed in a circle are congruent, if their vertex angles are equal.



Ex. 10.



Ex. 14.



Ex. 15.

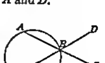
14. If FM is an altitude of inscribed $\triangle FGH$, and FL is a diameter, prove that $\triangle FGM$ and FLH are mutually equiangular.

15. If AB and CD are perpendicular diameters and E is any point on $\widehat{AC}$, prove that ED bisects $\angle AEB$.

16. Two equal chords produced meet outside the circle. Prove that the secants thus formed are equal.

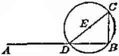
17. If $ABCDE$ is an inscribed pentagon (five sides) and $\widehat{BC} = 70°$, find the number of degrees in the sum of angles A and D.

18. An isosceles triangle, whose equal angles are each 72°, is inscribed in a circle. Prove that its base equals the side of an equilateral pentagon inscribed in the same circle.

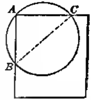


19. Two secants AC and DE intersect on the circle at B. Prove that $\angle ABD$ is measured by one-half of $\widehat{ABE}$.

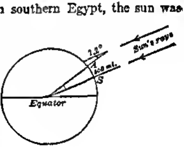
40. To erect a perpendicular to a line AB at the end B, take any point E over the line as a center and EB as a radius, and draw a circle cutting the line at B and again at D. Draw the diameter DC, and join the points C and B. Prove that CB is the required perpendicular.



41. If a rectangular piece of paper is placed on a circle as shown, prove that BC is the diameter. Draw a circle and find its center by the method suggested in the preceding sentence.



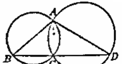
42. Eratosthenes, who lived in 200 B.C., determined the circumference of the earth as follows: He noted that at Syene in southern Egypt, the sun was directly overhead when farthest north, while at Alexandria, 500 miles farther north, the shadow of a vertical pole made an angle of 7.2° with the pole. Therefore he decided that the arc from Syene to Alexandria was one-fiftieth the circumference of the earth. Prove that he was right, and find the circumference of the earth from his data.



Investigation Problem. Draw a line BC tangent to a circle O at point B. Now draw a chord AB making an obtuse angle with BC. How do you think the number of angle degrees in $\angle ABC$ compares with the number of arc degrees in the major arc AB? To test your conclusion, draw the diameter BD. What is $\angle ABD$ measured by? How many degrees are there in $\angle DBC$? Why? How many degrees are there in $\widehat{DB}$? Why? How do they compare? What then is $\angle ABC$ measured by? Complete the proof. What change would you make in your proof, if $\angle ABC$ were acute? Do you think that any angle whose vertex is on the circle is measured by half of some arc? Try to prove some other case.

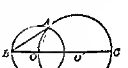
Honor Work

32. If AB and AD are diameters, prove that BD passes through point C.



33. An octagon $ABCDEFGH$ (eight sides) is inscribed in a circle. Prove that $\angle A + \angle C + \angle E + \angle G =$ three straight angles.

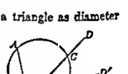
34. If two equal circles intersect in A and C, and any line BD is drawn through C ending in the circles, A is equally distant from its ends.



35. An exterior angle of an inscribed quadrilateral equals the opposite interior angle.

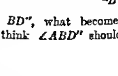
36. If $\odot O'$ passes through the center of $\odot O$, prove that $\angle ABC$ is measured by $\frac{1}{2}\widehat{AC}$.

37. A circle, constructed on one side of a triangle as diameter passes through the feet of the altitudes to the other two sides.



38. If BD rotates around point B toward the position of the tangent BD'', how does point C move? How does $\widehat{AC}$ change? Before BD arrives at the position BD'', what are is $\angle ABD$ measured by? When BD arrives at the position BD'', what becomes of this arc? What arc, then, would you think $\angle ABD''$ should be measured by?

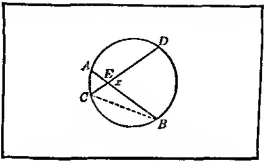
Applied Problems



39. When a carpenter wishes to learn whether a semicircle is accurately cut, he places his square so that the sides touch at A and C. If B just touches the curve in every position, he knows that the cut is semicircular. Why is this true?

Proposition 9

171. *An angle formed by two chords intersecting inside a circle is measured by half the sum of the opposite intercepted arcs.*



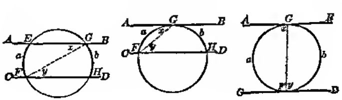
Given: $\angle x$ formed by chords AB and CD intersecting at E inside the circle.

To prove: $\angle x$ is measured by $\frac{1}{2}(\widehat{AC}+\widehat{BD})$.

Proof:

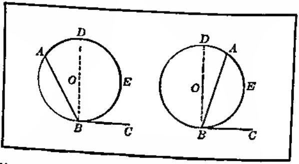
Statements	Reasons
1. Draw BC.	1. Post. 1.
2. $\angle x = \angle B + \angle C$.	2. § 87.
3. $\angle B$ is measured by $\frac{1}{2}\widehat{AC}$ and $\angle C$ is measured by $\frac{1}{2}\widehat{BD}$.	3. § 165.
4. $\angle x$ is measured by $\frac{1}{2}(\widehat{AC}+\widehat{BD})$.	4. Ax. 1.

172. Corollary. *Parallel lines intercept equal arcs on a circle.*



Proposition 8

170. *An angle formed by a tangent and a chord is measured by half its intercepted arc.*



Given: [⊙O]; chord AB meeting tangent BC at B.

To prove: $\angle ABC$ is measured by $\frac{1}{2}\widehat{AEB}$.

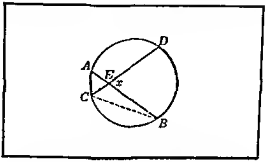
Proof:

Statements	Reasons
1. Draw diameter BD.	1. Post. 1.
2. $BD \perp BC$ and $\angle DBC$ is a rt. angle.	2. § 152.
3. $\widehat{DEB}$ is a semicircle.	3. § 143.
4. $\angle DBC$ is measured by $\frac{1}{2}\widehat{DEB}$.	4. § 164.
5. $\angle ABD$ is measured by $\frac{1}{2}\widehat{AD}$.	5. § 165.
6. $\angle ABC$ is measured by $\frac{1}{2}\widehat{AEB}$.	6. Ax. 3 or 4.

Investigation Problem. Draw a circle with two chords AB and CD crossing at E. Do you think that $\angle AEC$ is measured by a definite fraction of $\widehat{AC}$? Can you draw another figure in which $\angle AEC$ is the same size as it is in the figure you drew first, but in which $\widehat{AC}$ is larger? When $\widehat{AC}$ becomes larger, what change do you find in $\widehat{BD}$? Might not $\angle AEC$ be measured by a part of both arcs? What proposition comparing angles with arcs might be used here? Draw BC. How does $\angle AEC$ compare with $\angle C$ and $\angle B$? Why? What arcs are these angles measured by? What then is $\angle AEC$ measured by?

Proposition 9

171. *An angle formed by two chords intersecting inside a circle is measured by half the sum of the opposite intercepted arcs.*

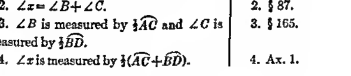


Given: $\angle x$ formed by chords AB and CD intersecting at E inside the circle.

To prove: $\angle x$ is measured by $\frac{1}{2}(\widehat{AC}+\widehat{BD})$.

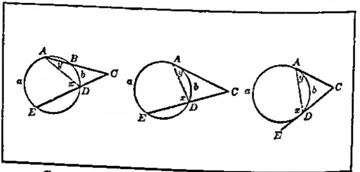
Proof: Statements	Reasons
1. Draw BC.	1. Post. 1.
2. $\angle x = \angle B + \angle C$.	2. § 87.
3. $\angle B$ is measured by $\frac{1}{2}\widehat{AC}$ and $\angle C$ is measured by $\frac{1}{2}\widehat{BD}$.	3. § 165.
4. $\angle x$ is measured by $\frac{1}{2}(\widehat{AC}+\widehat{BD})$.	4. Ax. 1.

172. Corollary. *Parallel lines intercept equal arcs on a circle.*



Proposition 10

173. *An angle formed by two lines intersecting outside a circle, and which meet the circle, is measured by half the difference of its intercepted arcs.*



Case 1. Both secants. Case 2. Secant and tangent. Case 3. Both tangents.

Given: AC and EC intercepting arcs a and b.

To prove: $\angle C$ is measured by $\frac{1}{2}(\widehat{a}-\widehat{b})$.

Proof:

Statements	Reasons
1. Draw AD.	1. Post. 1.
2. $\angle C+\angle y=\angle x$.	2. § 87.
3. $\angle C=\angle x-\angle y$.	3. Ax. 4.
4. $\angle x$ is measured by $\frac{1}{2}\widehat{a}$.	4. § 165 or § 170.
5. $\angle y$ is measured by $\frac{1}{2}\widehat{b}$.	5. § 165 or § 170.
6. $\angle C$ is measured by $\frac{1}{2}(\widehat{a}-\widehat{b})$.	6. Ax. 1.

Class Exercises

1. If two intersecting chords divide a circle into arcs whose lengths in order are 3 in., 4 in., 6 in., and 2 in., find the angle made by the chords.

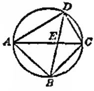
2. The line joining the points 8 and 11 on the clock is perpendicular to the line joining the points 4 and 9.

3. An angle formed by a tangent and a chord equals the angle inscribed in the opposite segment.

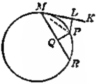
4. A tangent through the vertex of an inscribed isosceles triangle is parallel to the base.

5. Three consecutive sides of an inscribed quadrilateral cut off arcs of 65°, 95°, and 125°. Find the angles of the quadrilateral and the angle formed by its diagonals.

6. If $\widehat{AD}=114°$, $\widehat{AB}=106°$, and $\angle AEB=80°$, find the other angles of the figure.



Exs. 6, 7.



Ex. 8.

7. If $\widehat{AB}=35°$, $\widehat{BC}=103°$, and $\angle AED=119°$, how many degrees are there in each angle of the quadrilateral?

8. If, from a point on a circle, a chord and a tangent are drawn, the perpendiculars to them from the middle point of the arc are equal.

9. An angle formed by a tangent and a chord is the supplement of any angle inscribed in the segment cut off by the chord.

10. How many degrees are there in the angle formed by the side of an inscribed square and a tangent through one of the adjacent vertices?

11. If two chords intersect at right angles, the sum of a pair of opposite intercepted arcs equals a semicircle.

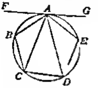
12. If the diagonals AC and BG of the equilateral inscribed octagon $ABCDEFGH$ intersect at K, how many degrees are there in $\angle AKB$?

13. The tangents through the vertices of an inscribed isosceles triangle form another isosceles triangle.

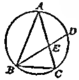
Optional Exercises

14. $ABCDE$ is an inscribed equilateral pentagon. Prove that $\angle A$ is trisected by AC and AD.

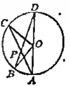
15. Using the same hypothesis, prove that the tangent at A is parallel to CD.



Exs. 14, 15.



Ex. 18.



Ex. 21.

16. Compare the perimeters of two equilateral triangles, one inscribed in a circle, the other circumscribed about the same circle.

17. AB is the diameter of a circle; AC and BD are chords intersecting at E, and $\angle AED = 40°$. Find the number of degrees in $\widehat{DC}$.

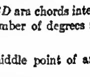
18. If $AB = AC$, $\angle A = 30°$, and D is the middle point of arc AC, find $\angle AED$.

19. Two intersecting chords divide a circle into arcs, proportional in order to 1, 2, 3, and 4. Find the angle formed by the chords.

20. KL and MP are two mutually perpendicular chords. If $\widehat{KM} = 58°$, and $\widehat{ML} = 108°$, find the number of degrees in $\angle KLP$.

21. If, from the ends of a diameter AD of $\odot O$, two chords AC and DB are drawn intersecting at P so as to make $\angle APB = 45°$, then $\angle BOC$ is a right angle.

Quadrilateral $ABCD$ is inscribed in a circle. Find the number of degrees in $\angle DAB$ and in $\angle ABC$, if.

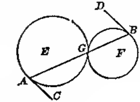


22. $\widehat{BC}$ is twice $\widehat{AB}$, $\widehat{CD}$ is 15° more than $\widehat{BC}$ and $\widehat{AD}$ is 20° less than $\widehat{CD}$.

23. Arcs AD, DC, and CB are equal, and each 10° more than $\widehat{BA}$.

Honor Work

In the exercises in which two circles are tangent to each other, draw the common internal tangent.

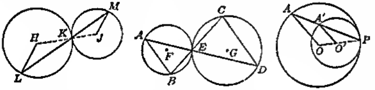


24. Circles E and F are tangent at G. Prove that the tangents AC and BD are parallel.

25. If $\odot H$ and $\odot J$ are tangent at K, prove that LH is parallel to JM.

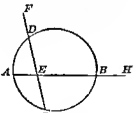
26. If $\odot F$ and $\odot G$ are tangent at E, prove AB parallel to CD.

27. Circles O and O' are tangent internally at P. Prove that OA is parallel to $O'A'$.

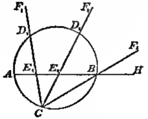


Ex. 25. Ex. 26. Ex. 27.

28. In the first figure, two chords AB and CD intersect at E. What is $\angle AEC$ measured by? What is $\angle AEF$ measured by? If the



Ex. 28.

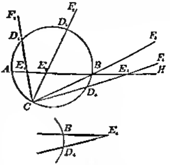


Ex. 28.

point E moves along AB toward B while A, B, and C remain fixed, what change takes place in $\widehat{AD}$? In $\widehat{DB}$? When E arrives at B, what angle has $\angle AEC$ become? $\angle AEF$? What arc has DB become? AD? What, then, is $\angle ABC$ measured hy? $\angle ABF_3$?

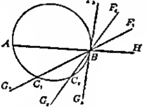
29. If line CF does not stop when $\widehat{DB}$ becomes 0° but continues to turn around point C until E is outside the circle at E_4, where now is $\widehat{DB}$? If a quantity decreases to zero and then continues in the same direction, what would you expect its value to become? What is $\angle AE_4C$ measured by?

NOTE. Notice, in the above examples, that the arc is negative when it bends toward the vertex of the angle.

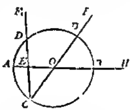


Ex. 29.

30. Returning to the point where E coincides with B, what is $\angle ABG_1$ measured by? $\angle ABF_1$? If now GF turns around point B, as in the figure, how does C move? When GF becomes the tangent G_4F_4, what has $\widehat{AC}$ become? $\widehat{CB}$? What is $\angle ABG_4$ measured by? $\angle ABF_4$?



Ex. 30.



Ex. 31.

31. Returning again to the original intersecting chords, by what arcs is $\angle AEC$ measured? If AB becomes a diameter and point E moves to the center O, what do these arcs become? What then is $\angle AOC$ measured by? Reconcile this with § 162.

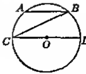
32. Considering an arc negative when it bends toward the vertex of the angle, and zero when its ends coincide, prove that: *Any angle, whose sides meet a circle, is measured by one-half the algebraic sum of its intercepted arcs.*

Class Exercises

1. If $AB \parallel CD$, CD is a diameter, and $\widehat{AB}=80°$, find $\angle C$.

2. If $AB \parallel CD$, CD is a diameter, and $\angle C=20°$, find $\widehat{AB}$.

3. If CD is a diameter, $\widehat{AC}=40°$, and $\widehat{AB}=90°$, is $AB \parallel CD$? Why?



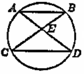
Exs. 1, 2, 3.

4. An inscribed trapezoid is isosceles.

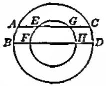
5. If two arcs of a circle, with no common point, are equal, their ends are the vertices of an isosceles trapezoid or rectangle.

6. If a tangent through the vertex of an inscribed triangle is parallel to the base, the triangle is isosceles.

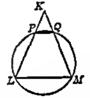
7. If $AB \parallel CD$, then $AD=BC$.



Exs. 7, 8.



Ex. 9.



Exs. 10, 11.

8. If $AB \parallel CD$, then $\triangle CED$ is isosceles.

9. If $\odot ABDC$ and $\odot EFHG$ have the same center, and $\widehat{AB}=\widehat{CD}$, then $\widehat{EF}=\widehat{GH}$.

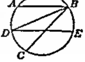
Optional Exercises

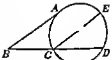
10. If KLM is an isosceles triangle, prove that $PQ \parallel$ base LM.

11. If $PQ \parallel LM$, prove that $\triangle KLM$ is isosceles.

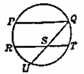
12. If the side AD of inscribed quadrilateral $ABCD$ is less than side BC, then AB is not parallel to CD.

13. If DB bisects $\angle ABC$ and $DE \parallel AB$, prove that $DE=BC$.





14. AB is a tangent and CE is parallel to AB. Show that $\angle B$ is measured by one-half the difference of arcs AD and AC, by means of the relation $\widehat{ED} = \widehat{AD} - \widehat{AE}$.

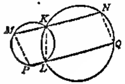


15. If $PQ \parallel RT$, prove that $\angle RSU$ is measured by $\frac{1}{2}(RU + QT)$, by showing that $PU = RU + QT$.

Honor Work

16. The common external tangents to two equal circles are parallel and equal.

17. Two circles cut at K and L. If parallel lines MKN and PLQ are drawn, prove that MN equals PQ. (Prove MP parallel to NQ.)



Ex. 17.

A Self-Measuring Test

1. Complete the following statements:

(a) A central angle is measured by .....

(b) An inscribed angle is measured by .....

(c) An angle formed by a tangent and chord is measured by .....

(d) An angle formed by two chords crossing inside a circle is measured by .....

(e) An angle formed by two secants meeting outside a circle is measured by .....

(f) An angle, whose vertex is inside the circle, is measured by .....

(g) An angle, whose vertex is on the circle, is measured by .....

(h) An angle, whose vertex is outside the circle, is measured by .....

(i) An angle inscribed in a semicircle is .....

2. Make a list of ten methods of proving angles equal.

3. Give three propositions about a tangent.

4. Give three methods of proving arcs equal.

5. What proposition on the measurement of angles by arcs depends on the fact that the base angles of an isosceles triangle are equal?

6. What propositions on the measurement of angles by arcs depend on the fact that the exterior angle of a triangle equals the sum of the remote interior angles?

7. State two propositions about equal chords; about equal arcs; about a radius and a line perpendicular to it.

8. Give a method, dependent on the circle of proving that an angle is a right angle.

9. Is a central angle of a circle measured by half the sum of two arcs?

10. Explain what is meant by the statement that an angle is measured by half its arc.

11. What part of a circle is intercepted by an inscribed right angle? By an inscribed acute angle?

12. *Is a line perpendicular to a radius always a tangent?*

13. State a proposition about parallel lines cutting a circle. Is the converse of this proposition true?

14. What is the point where a tangent touches a circle called?

15. Where is the vertex of an inscribed angle? What are its sides?

16. Define: *chord; arc; diameter; tangent; secant.*

17. Can secants intercept equal arcs on a circle, if they meet outside the circle? If they meet inside the circle?

Class Exercises

Trace back each of the following propositions through two generations, that is, give *all propositions used as reasons* in the proof and all propositions used in proving those propositions.

1. A radius perpendicular to a chord bisects the chord and its arc.

2. An inscribed angle is measured by half its intercepted arc.

3. If three or more parallel lines cut off equal lengths on one transversal, they cut off equal lengths on every transversal.

4. Two right triangles are congruent if the hypotenuse and a leg of one equal the hypotenuse and a leg of the other.

5. A diagonal of a parallelogram divides it into two congruent triangles.

6. In a circle or in equal circles, equal chords are equally distant from the center.

Honor Work

Trace each of the following propositions back to axioms or definitions giving all propositions, definitions, axioms, and postulates on which they depend.

7. An angle formed by two chords intersecting inside a circle is measured by half the sum of the opposite intercepted arcs.

8. In a circle equal chords have equal arcs.

9. Any point on the bisector of an angle is equally distant from the sides of the angle.

10. The perpendicular bisector of a given line segment can be constructed.

Numerical Test (*10 min.*)

1. $\triangle ABC$ is inscribed in a circle. If $\angle A = 40°$ and the exterior angle at C is $150°$, find arc AC.

2. AB is a diameter and AK a chord in a circle such that $\angle BAK = 55°$. Find the number of degrees in arc AK.

3. $\triangle ABC$ is inscribed in a circle. If $\angle A = 42°$ and $\angle B = 68°$, find the number of degrees in the minor arc AB.

4. If $\angle P$, formed by two tangents PA and PB drawn to circle O, is $70°$, and if OA and OB are drawn, find the number of degrees in $\angle AOB$.

5. The diameter DC of a circle is perpendicular to chord AB. If $\widehat{AC} = 70°$, find the number of degrees in $\widehat{BD}$.

Completion Test (*10 min.*)

1. In a circle an angle inscribed in an arc that is less than a semicircle is an . . . angle.

2. AB is a diameter of a circle and AC is a chord such that $\angle BAC = 30°$. If AB is 12 in. long, then chord BC is . . . in. long.

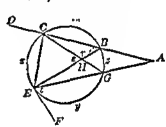
3. If a central angle and the angle formed by a tangent and a chord intercept the same arc, the ratio of the angles is . . . .

4. The sides of a triangle inscribed in a circle cut off arcs that have the ratio $3:4:5$. The number of degrees in the largest angle of the triangle is . . . .

5. AB is a diameter of a circle, AC is a chord and $\overset{\frown}{AC} = 110°$. Then $\angle BAC =$ . . . degrees.

Test on Measurement of Angles (*10 min.*)

In exercises 1 to 6, $x = 90°$, $y = 130°$, $z = 40°$ and EF is a tangent.



1. $\angle r =$ . . . .
2. $\angle s =$ . . . .
3. $\angle t =$ . . . .
4. $\angle CEG =$ . . . .
5. $\angle A =$ . . . .
6. $\angle DCE =$ . . . .
7. If $\angle r = 35°$, then $x =$ . . . .
8. If $\angle A = 15°$ and $z = 35°$, then $x =$ . . . .

Matching Test (*10 min.*)

After each number write the letter of the phrase that corresponds.

1. Inscribed angle	*a.* Measured by its arc.
2. An angle of 25° formed by a tangent and chord	*b.* Measured by half the difference of its arcs.
3. An angle formed by intersecting chords	*c.* Measured by half the sum of its arcs.
4. An obtuse angle	*d.* Measured by half its arc.

2. An inscribed angle is measured by half its intercepted arc.

3. If three or more parallel lines cut off equal lengths on one transversal, they cut off equal lengths on every transversal.

4. Two right triangles are congruent if the hypotenuse and a leg of one equal the hypotenuse and a leg of the other.

5. A diagonal of a parallelogram divides it into two congruent triangles.

6. In a circle or in equal circles, equal chords are equally distant from the center.

HONOR WORK

Trace each of the following propositions back to axioms or definitions giving all propositions, definitions, axioms, and postulates on which they depend.

7. An angle formed by two chords intersecting inside a circle is measured by half the sum of the opposite intercepted arcs.

8. In a circle equal chords have equal arcs.

9. Any point on the bisector of an angle is equally distant from the sides of the angle.

10. The perpendicular bisector of a given line segment can be constructed.

NUMERICAL TEST *(10 min.)*

1. $\triangle ABC$ is inscribed in a circle. If $\angle A = 40°$ and the exterior angle at C is $150°$, find arc AC.

2. AB is a diameter and AK a chord in a circle such that $\angle BAK = 55°$. Find the number of degrees in arc AK.

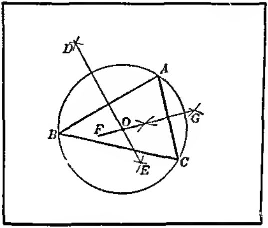
3. $\triangle ABC$ is inscribed in a circle. If $\angle A = 42°$ and $\angle B = 68°$, find the number of degrees in the minor arc AB.

4. If $\angle P$, formed by two tangents PA and PB drawn to circle O, is $70°$, and if OA and OB are drawn, find the number of degrees in $\angle AOB$.

5. The diameter DC of a circle is perpendicular to chord AB. If $\widehat{AC} = 70°$, find the number of degrees in $\widehat{BD}$.

PROPOSITION 11

*** 174.** *A circle can be circumscribed about a given triangle.*



Given: $\triangle ABC$.

To prove: A circle can be circumscribed about $\triangle ABC$.

Construction: STATEMENTS	REASONS
1. Construct DE the $\perp$ bisector of AB, and FG the $\perp$ bisector of AC, meeting in O.	1. § 63.
2. With O as center and OA as radius, construct a $\odot$. $\odot O$ is the required circle.	2. Post. 3.

Proof:

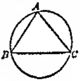
1. $OA = OB$ and $OA = OC$.	1. § 59.
2. $\odot O$ passes through A, B, and C.	2. § 140.
3. $\odot O$ is circumscribed about $\triangle ABC$.	3. § 136.

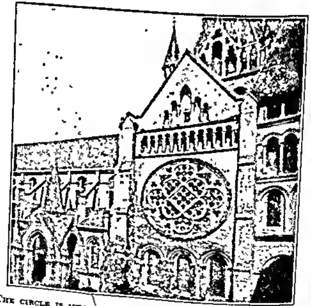
If the perpendicular bisector of BC were drawn, would its intersection with FO give the center of another circle through A, B and C?

5. A central angle
6. An angle outside a circle
7. An acute angle
8. An inscribed angle of 50°
9. A central angle of 25°
10. A right angle

e. Measured by half a semicircle.
f. Intercepts an arc of 50°.
g. Intercepts an arc of 100°.
h. Intercepts an arc of 25°.
i. Inscribed in a minor arc.
j. Inscribed in a major arc.

Investigation Problem. To construct a circle through the points A, B, and C, I must find a point O which is .. . . . from these three points. All points equally distant from A and B are on the .... .. of .... .. All points . ... from A and C are on the . .... of . The intersection of these two lines will be .... from A, B, and C.

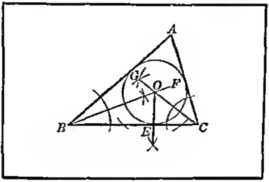




THE CIRCLE IS MUCH USED IN THE DESIGN OF CHURCH WINDOWS. This is the cathedral at Lausanne, Switzerland.

Proposition 12

177. *A circle can be inscribed in a given triangle.*



Given: $\triangle ABC$.

To prove: A circle can be inscribed in $\triangle ABC$.

Construction: Statements	Reasons
1. Construct BF bisecting $\angle B$, and CG bisecting $\angle C$, and meeting in O.	1. § 65.
2. From O, construct $OE \perp BC$.	2. § 64.
3. With O as center and OE as radius, construct $\odot O$.	3. Post. 3.

$\odot O$ is the required circle.

Proof:

1. O is equally distant from AB, BC, and AC.	1. § 97.
2. BC, AB, and AC are tangents.	2. § 153.
3. $\odot O$ is inscribed in $\triangle ABC$.	3. § 137.

1. By extending the sides of the triangle, construct other circles touching all three sides.

2. If $\angle A$ grows larger, how does $\angle BOC$ change? When $\angle A$ has increased 40°, how many degrees has $\angle BOC$ changed?

*** 175. Corollary 1.** *Through three points not on a straight line, one and only one circle can be constructed.*

Let A, B and C, in the figure of Proposition 11, be the three points. If the circle passes through A and C, must its center be on FG, the perpendicular bisector of AC? Why? (§ 60). If the circle also passes through A and B, on what line must the center lie? How many points fulfil both of these conditions? (§ 5). Is point O also on the perpendicular bisector of BC? (§ 60).

176. Corollary 2. *Two circles can intersect in only two points.*

Exercises

1. Where is point O, in the above figure, (a) when $\angle A$ is acute? (b) When $\angle A$ becomes a right angle? (c) When $\angle A$ becomes obtuse? (d) When A approaches very near to BC? (e) When A is on BC? (f) When A crosses to the other side of BC?

2. How do the positions of ED and FG change for each of these cases? When A reaches BC, what is then true of ED and FG? Where do they intersect when A crosses to the other side of BC?

3. How does the size of $\odot O$ change as A moves nearer to line BC? What becomes of the circle when A moves on to BC?

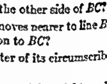
4. If $\triangle ABC$ is acute, prove that the center of its circumscribed circle is inside the triangle.

5. State and prove a similar exercise when $\angle A$ is a right angle. An obtuse angle.

A ... straight line can intersect a circle in only two points.

... tion Problem. To construct a circle touching AB, BC and AC, ...

... vestiga... must find a point O .... from t... these .... All points equally distant ... from AB and BC are on the ..... of ... . All points equally distant from AC ... nd BC are on the ..... of ..... The int... ersection of these two lines will be equally distant from ....., ... and .....



CASE 2. When P is outside the circle.

Construction:

1. Construct OP.	1. Post. 1.
2. Bisect OP at B.	2. § 63.
3. With B as center and OB as radius construct a $\odot$ cutting $\odot O$ at A and C.	3. Post. 3.
4. Draw AP and CP.	4. Post. 1.

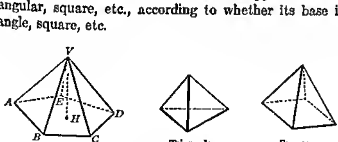
Then AP and CP are the required tangents.

Proof:

1. Draw OA and OC.	1. Post. 1.
2. $AP \perp OA$ and $CP \perp OC$.	2. § 166.
3. AP and CP are tangent to $\odot O$.	3. § 153.

SPACE GEOMETRY *(Optional)*

A **pyramid** is a solid figure of which one face, the **base**, is a polygon having any number of sides and the other faces are triangles meeting at a common vertex. The faces meeting at the common vertex are the **lateral faces**, as VAB, VBC, etc. The **altitude** VH of the pyramid is the perpendicular from the vertex to the plane of the base. A pyramid is called triangular, square, etc., according to whether its base is a triangle, square, etc.



Triangular Square

EXERCISES

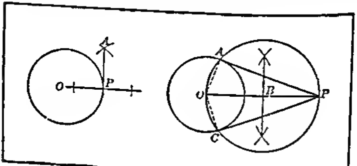
1. Can a sphere be circumscribed about any triangular pyramid? Prove your answer.

2. Can a sphere be inscribed in any triangular pyramid? Prove your answer.

3. If $\angle A$ grows larger, but the size of $\odot O$ remains unchanged, describe the change of position of points B and C.

PROPOSITION 13

178. *Through a given point, on or outside a circle, a tangent to the circle can be constructed.*



CASE 1 CASE 2

Given: Point P, on or outside $\odot O$.

To prove: A tangent to $\odot O$ through P can be constructed.

CASE 1. When P is on the circle.

Construction:

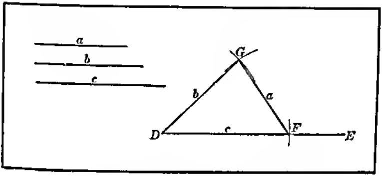
STATEMENTS	REASONS
1. Construct OP and extend it outside $\odot O$.	1. Post. 1 and 2.
1. Construct $PA \perp OP$ at P.	2. § 64.
PA is the required tangent.	

Proof:

1. $PA \perp OP$.	1. Const.
2. PA is tangent to $\odot O$.	2. § 153.

Proposition 14

180. *Given its three sides, a triangle can be constructed.*



Given: Line segments a, b, and c, the sides of a triangle.

To prove: The triangle can be constructed.

Construction:

Statements	Reasons
1. With D as center and c as radius, cut DE at F.	1. Post. 3.
2. With D as center and b as radius, construct an arc.	2. Post. 3.
3. With F as center and a as radius, cut the last arc at G.	3. Post. 3.
4. Draw DG and FG.	4. Post. 1.

$\triangle DFG$ is the required triangle.

Proof:

1. The sides of $\triangle DFG$ are a, b, and c.	1. Const.

Is it possible to choose the lengths of the segments a, b and c so that no triangle could be constructed having those segments as its sides? Explain your answer.

Is it possible to construct another triangle, having a different shape or size from $\triangle DFG$, with the same three segments as sides? Prove your answer.

3. Discuss the number of spheres that can be passed through 2 points; 3 points; 4 points; 5 points.

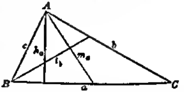
4. What is the relation of a radius of a sphere to the plane tangent to the sphere at its end?

5. If two planes are tangent to a sphere at opposite ends of a diameter, what is their relation to each other?

6. Is it possible to have more than two lines tangent to a sphere from the same outside point?

7. What can you say about the location of all lines tangent to a sphere at a given point on the sphere?

179. Notation. The vertices of a triangle are denoted by the capital letters A, B, and C. The opposite sides are denoted by the small letters a, b, and c; a being opposite $\angle A$, b opposite $\angle B$, etc.



The altitude from A is denoted by h_a, etc., the median from A by m_a, etc., and the bisector of $\angle A$ by t_a, etc.

EXERCISES

1. In an equilateral triangle, the points of contact of the inscribed circle bisect the sides.

2. A triangle is equilateral if the points of contact of its inscribed circle bisect its sides.

3. If the sides of a circumscribed quadrilateral are all bisected by their points of contact, the quadrilateral is equilateral.

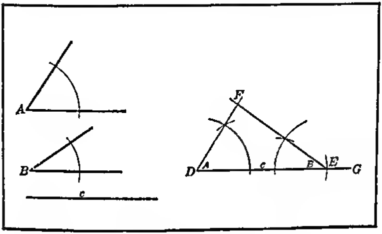
4. In the diagram of Proposition 12, if $\angle A = 80°$, find the number of degrees in $\angle BOC$.

5. In Marion Bradshaw's garden three paths meet to form a triangle. Marion wishes to lay out a circular flower bed which will touch all three walks. Construct the plan for it.

6. Construct a plan for a running track whose ends are semicircles and whose sides are straight lines tangent to the curves.

Proposition 16

182. *Given twa ongles whose sum is less than a straight angle and the included side, a triangle can be constructed.*



Given: $\angle A$ and $\angle B$ whose sum is less than a straight angle, and line segment c.

To prove: A triangle can be constructed with two angles and the included side equal to A, B, and c, respectively.

Construction: Statements	Reasons
1. With D as center and c as radius, cut DG at E.	1. Post. 3.
2. At D construct an angle equal to $\angle A$.	2. § 66.
3. At E construct an angle equal to $\angle B$.	3. § 66.
$\triangle DEF$ is the required triangle.	

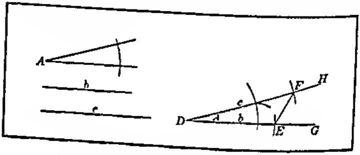
Proof:

1. Two angles and the included side equal A, B, and c, respectively.	1. Const.

Is it possible to choose the given parts so that no triangle could be constructed?

Proposition 15

181. *Given two sides and the included angle, a triangle can be constructed.*



Given: $\angle A$, line segments b and c.

To prove: A triangle can be constructed with two sides and the included angle equal to b, c, and A respectively.

Construction:

Statements	Reasons
1. With D as center and b as radius, cut DG at E.	1. Post. 3.
2. Construct $\angle D = \angle A$.	2. § 66.
3. With D as center and c as radius, cut DH at F.	3. Post. 3.
4. Draw EF.	4. Post. 1.
$\triangle DEF$ is the required triangle.	

Proof:

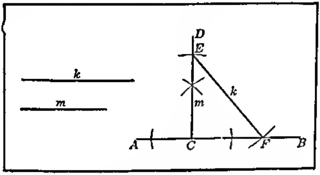
1. Two sides and the included angle equal b, c, and A respectively.	1. Const.

Is it possible to choose the given parts in proposition 15 so that no triangle could be constructed?

Is it possible to construct, with these same given parts, another triangle differing in size or shape from $\triangle DEF$? Prove your answer.

Proposition 18[1]

184. *Given the hypotenuse and a leg, a right triangle can be constructed.*



Given: Line segments k and m, with k longer than m.

To prove: A right triangle can be constructed with hypotenuse k and leg m.

Construction: Statements	Reasons
1. Construct $CD \perp AB$ at any point C.	1. § 64.
2. With C as center and m as radius, cut CD at E.	2. Post. 3.
3. With E as center and k as radius, cut AB at F.	3. Post. 3.
4. Draw EF.	4. Post. 1.

Then CEF is the required triangle.

Proof:

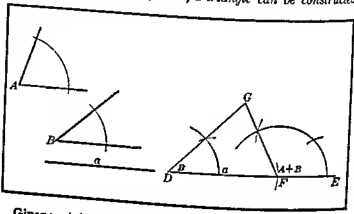
1. The hypotenuse and leg equal k and m respectively, and $\angle C$ is a rt. angle.	1. Const.

Is it possible to so choose the given parts that no triangle can be constructed?

[1] This proposition is optional. It is not mentioned by the National Committee on Mathematical Requirements or by the College Entrance Examination Board.

Proposition 17

183. *Given two angles whose sum is less than a straight angle and a side opposite one of them, a triangle can be constructed.*



Given: $\angle A$ and $\angle B$ whose sum is less than a straight angle, and line segment a.

To prove: A triangle can be constructed with two angles and a side opposite one of them equal to A, B, and a respectively.

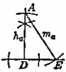
Construction:

Statements	Reasons
1. With D as center and a as radius, cut DE at F.	1. Post. 3.
2. At D construct an angle equal to $\angle B$.	2. § 66.
3. At F construct an angle equal to $\angle B + \angle A$.	3. § 66.
$\triangle DFG$ is the required triangle.	

Proof:

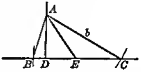
1. $\angle D = \angle B$ and $DF = a$.	1. Const.
2. $\angle GFE = \angle B + \angle A$.	2. Const.
3. $\angle GFE = \angle D + \angle G$.	3. § 87.
4. $\angle D + \angle G = \angle B + \angle A$.	4. Ax. 2.
5. $\angle G = \angle A$.	5. Ax. 4.

STEP 2. Examine this figure. We observe that $\triangle ADE$ is a right triangle whose hypotenuse and leg are known.



STEP 3. Construct $\triangle ADE$.

STEP 4. Examining the first figure again, we notice that C can be found by taking A as center and b as radius, and cutting DE at C. Finally, since AE is a median, and BE equals EC, B can be found and the triangle completed. (NOTE: A different triangle would have been constructed with the same parts, if AC cut DE on the other side of D.)



From these two illustrations, we observe that the method of discovering the solution of a construction problem is as follows:

1. Draw a figure like the one you wish to construct, using solid lines for the given parts and dotted lines for the unknown parts, as in the illustrations given.
2. Examine this figure to see if there is any part of it which you can construct. You have at your command five methods of constructing triangles, Propositions 14 through 18. Find a triangle in your figure which has enough parts known so that you can use one of these five.
3. Construct that part, disregarding the rest of the figure.
4. Finally, build the rest of the figure around that part.

EXERCISES

1. Construct an equilateral triangle, given: (a) a side; (b) the altitude; (c) the perimeter.

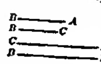
2. Divide a line into: (a) four equal parts; (b) eight equal parts.

3. Divide a circle into: (a) six equal parts; (b) three equal parts.

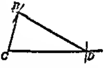
185. Analysis of a construction problem. Analysis is the method of discovering the solution of a construction problem. This method will be explained by the following examples.

ILLUSTRATION 1. Construct a trapezoid, given three sides and one diagonal.

STEP 1. Suppose the construction completed, and draw a figure $ABCD$ as you think the trapezoid should look. Make solid lines for the three given sides AB, BC, and CD, and for the diagonal BD, and a dotted line for the unknown side AD.



STEP 2. Examine this figure to see if there is a triangle which can be constructed by one of the known methods. We note that $\triangle BCD$ can be constructed as all three sides are known.



STEP 3. Construct $\triangle BCD$.

STEP 4. Now examine the first figure to discover the relation of the remainder of the trapezoid to this triangle. We note that BA is parallel to CD. Construct this parallel and measure the length BA on it. Then complete the trapezoid by drawing AD.

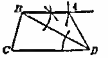
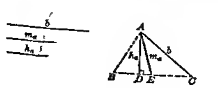


ILLUSTRATION 2. Construct a triangle, given b, m_a, and h_a.

STEP 1. Draw the completed figure, which should look like $\triangle ABC$, with AD, AE, and AC, the known lines, solid.



(g) One side, the two diagonals, and the angles which the diagonals make with the given side.

(h) The three sides AB, BC, and CD, the diagonal BD and $\angle ABD$.

13. Construct a parallelogram, given:

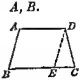
(a) Two adjacent sides and a diagonal.
(b) One side and the two diagonals.
(c) A side, a diagonal, and their included angle.
(d) A side, a diagonal, and the altitude on the given side.
(e) A side, an angle, and the altitude on the given side.

14. Construct a triangle, given:

(a) $a,\ b,\ h_a$.	(g) $A, B,\ h_a$.	(m) $A,\ h_a,\ h_b$.
(b) $a,\ b,\ h_c$.	(h) $A, B,\ h_c$.	(n) $A,\ h_b,\ h_c$.
(c) $a,\ b,\ m_a$.	(i) $A, B,\ t_a$.	(o) $A,\ h_a,\ t_a$.
(d) $A,\ b,\ h_a$.	(j) $a,\ h_b,\ m_b$.	(p) $A,\ h_b,\ t_a$.
(e) $A,\ b,\ h_b$.	(k) $a,\ h_b,\ h_c$.	(q) $a,\ h_a,\ h_b$.
(f) $A,\ b,\ t_a$.	(l) $a,\ h_a,\ m_a$.	(r) $A,\ B,\ t_c$.

15. Construct a triangle, given the radius of the circumscribed circle and:

(a) $a,\ b$.	(c) $a,\ m_a$.	(e) $a,\ h_b$.
(b) $A,\ b$.	(d) $a,\ h_a$.	(f) $A,\ B$.



16. Construct a trapezoid, given the four sides. (Construct $\triangle DEC$, then $\square EDAB$.)

17. How many circles are determined by four points, no three of which are in a straight line, if each circle passes through three of the points? Construct them.

18. In a given circle, construct two parallel chords which are equal.

19. In a given triangle, construct a semicircle tangent to two sides and having its diameter on the third side.

4. Construct eight points at equal distances apart and each 2 in. from a given point.

5. Construct $\triangle ABC$, given:

(a) $a=3$ in., $B=90°$, and $C=60°$.
(b) $a=3$ in., $B=60°$, and $C=45°$.

6. Construct an isosceles triangle, given:

(a) The base and a base angle.
(b) A leg and the vertex angle.
(c) The vertex angle and the altitude to the base.
(d) The vertex angle and the altitude to a leg.
(e) The vertex angle and the base.

7. Construct an isosceles right triangle, given: (a) a leg; (b) the hypotenuse.

8. Construct a square, given a diagonal.

9. Construct a rhombus, given a side and a diagonal.

10. Construct a right triangle, given:

(a) The legs.
(b) A leg and the opposite acute angle.
(c) The hypotenuse and an acute angle.
(d) The hypotenuse and a leg, by first laying off the hypotenuse and constructing a semicircle on it.
(e) The hypotenuse and the altitude on it. (Construct a semicircle on the hypotenuse and then a parallel to the hypotenuse.)
(f) The median and the altitude both on the hypotenuse.
(g) A leg and the altitude on the hypotenuse.

11. Construct a rectangle, given a side and a diagonal.

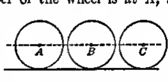
12. Construct a quadrilateral, given:

(a) Three sides and the two included angles.
(b) Four sides and one angle.
(c) Four sides and a diagonal.
(d) Three sides and the two diagonals.
(e) Three sides, one included angle, and the diagonal from that angle.
(f) $AB=BC=2$ in., $\angle A=120°$, $\angle B=90°$, and $\angle C=60°$.

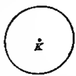
LOCI

186. So far in our work we have generally studied stationary figures. However, in life many of the objects with which we must deal are not stationary. We see around us automobiles, trains, and animals in motion. And even in machines which are themselves stationary, wheels are turning and rods are moving to and fro. Let us therefore consider some point on a moving object. How does it move and along what path does it travel?

Take, for example, the wheel of a car moving along a track. At one instant the center of the wheel is at A, a little later it is at B, then at C, and so on. Evidently it moves along a line passing through the points A, B, and C, and, if the track is straight, the path is a straight line parallel to the track and at a distance of the radius from it. Now, if instead of the wheel moving along a track, we consider the circle from a purely geometric standpoint, it is evident that it could move equally well on the other side of the line, and the center would trace another line parallel to the given line, and also at the distance of the radius from it. Thus in this case the path of our moving point would consist of two lines.



What is the path of a point which moves so that it is always three inches from a fixed point? Think of the fixed point as a nail in the wall, and of the moving point as the end of a piece of chalk fastened to the nail by a cord three inches long. Evidently the path is the circle whose center is the given point and whose radius is three inches. It is also evident that every point on this circle is exactly three inches from the



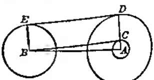
20. Construct a tangent to a given circle:

(a) Parallel to a given line.

(b) Perpendicular to a given line.

(c) Making a 60° angle with a given line.

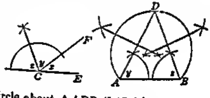
21. Construct a common external tangent to two given circles. (Take $AC = AD - BE$, and construct tangent BC.)



22. Construct the common internal tangent to two given circles.

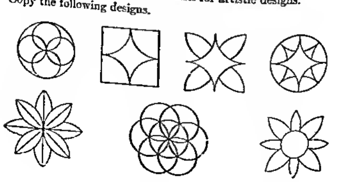
23. Through two given points, construct two parallel lines at a given distance d apart.

24. Construct a segment of a circle on AB as a chord, which will contain a given $\angle ECF$. [Bisect the supplement of $\angle ECF$ into angles y and z. Construct $\triangle ABD$ with $\angle y$, AB, and $\angle z$ as two angles and the included side (§ 182). Circumscribe a circle about $\triangle ABD$ (§ 174).]



Drawing Exercises

The circle is much used as the basis for artistic designs. Copy the following designs.



it, always keeping them in contact. What is the locus of the center of the cent?

8. Hold your ruler stationary on your desk and roll a cent along the edge, keeping them in contact. What is the locus of the center of the cent?

9. Katherine is walking through a field keeping equally distant from two straight intersecting roads. What is her path?

10. A cow is fastened to a stake with a rope 30 ft. long. If she walks, keeping the rope pulled tight, what is her locus?

187. To prove that a line is a locus, it is necessary to prove *two* propositions:

1. *That any point on the line satisfies the given condition.*

2. *Either* (*a*), *That any point which satisfies the given condition is on the line;*

or (*b*), *That any point not on the line does not satisfy the given condition.*

The first proves that a point can move along the line, and the second, that it cannot move off the line.

Ex. 1. Mark six points, each equally distant from two given points A and B. Do they lie on a straight line or on a circle? Are all points on this line equally distant from A and B? Can you find a point not on this line that is equally distant from A and B? What then is the locus of a point equally distant from two given points?

Ex. 2. Draw an $\angle ABC$. Select five points inside this angle, each the same distance from AB as from BC. What line does the locus appear to be? Is every point on this line equally distant from AB and BC? Is every point equally distant from AB and BC on this line? What then is the locus of a point equally distant from the sides of a given angle?

188. In § 59, it is proved that any point on the perpendicular bisector of a line segment is equally distant from the ends of the segment, and, in § 60, that any point equally distant from the ends of a line segment is on the perpendicular bisector of the segment. Therefore:

fixed point, and that every point three inches from the fixed point is on the circle.

The path of a point moving according to some given condition is called the **locus of the point.**

Locus (plural *loci*) is the Latin word meaning "place," and is related to the English words "locate" and "location." The locus of a point is the place where the point is. Instead of thinking of the locus as the path of a moving point, we sometimes think of it as the place where all the points are that fulfil the given condition, and we speak of it as the locus of points satisfying that condition.

Just as the point of the pencil, moving on the paper, traces a line, so the path of any moving point is a line, or, as we have seen above, a group of lines. All loci considered here are either straight lines or circles.

Exercises

1. Draw a straight horizontal line. Mark six points above the line, each 1 in. from it. Move the point of your pencil along the paper through these points keeping it always 1 in. from the given line. What is its path? Is there any point on this path that is not just 1 in. from the given line? Is there any point above the given line and 1 in. from it that is not on this path?

2. What is the locus of a point on this page one inch from the top of the page?

3. What is the locus of the outer end of the minute hand of a clock?

4. Robert's toy boat is drifting down the stream equally distant from the two banks. If the banks are straight and parallel, what is the locus of the boat?

5. Open and close the front cover of this book while it lies on the desk. What is the locus of the upper right-hand corner of the cover?

6. An automobile wheel is jacked up off the ground. If the wheel is made to spin, what is the locus of a point on one of the spokes?

7. Hold a nickel stationary on your desk and roll a cent around

10. What is the locus of the middle point of a chord of a given length that can be drawn in a given circle?

11. Two circles have the same center. Find the locus of the center of a circle tangent to both.

12. What is the locus of the centers of all circles which:

(a) Touch two intersecting lines?
(b) Pass through two given points?
(c) Touch a given line at a given point?
(d) Touch a given circle at a given point?

Optional Exercises

13. If a point so moves that the lines which connect it with two fixed points, A and B, are always perpendicular to each other, what is its locus?

14. The base BC of $\triangle ABC$ is fixed both in length and position. If the altitude from A equals a given line segment, what is tha locus of A?

15. A circle of radius $1\frac{1}{2}$ in. rolls around a square whose side is 2 in. Construct the locus of the center of the circle.

16. A circle whose radius is $\frac{1}{2}$ in. rolls on the inside of a square whose side is 2 in. Construct the locus of the center of the circle.

17. Construct the locus of the center of a circle of radius 1 in. which rolls around an equilateral triangle whose altitude is 2 in.

18. What is the locus of the center of a circle of radius 1 in. that rolls inside a rectangle whose length is 5 in. and whose width is 2 in.?

19. Within a rectangle, not a square, circles are drawn tangent to two sides of the rectangle. Find the locus of their centers.

Honor Work

20. What is the locus of the middle point of a line segment from a given point P and ending in the circumference of a circle, if:

(a) P is on the circle?
(b) P is outside the circle?
(c) P is inside the circle?
(d) P is the center of the circle?

* 189. *The perpendicular bisector of a line segment is the locus of a point equally distant from the ends of the segment.*

* 190. Similarly, from § 97 and § 98:

The bisector of an angle is the locus of a point equally distant from the sides of the angle.

Class Exercises

1. What is the locus of a point equally distant from two given points?

2. What is the locus of a point equally distant from two intersecting lines?

3. What is the locus of a point $\frac{1}{2}$ in. from a given straight line?

4. Describe the locus of a point:

   (a) Outside a given circle and $\frac{1}{2}$ in. from it.

   (b) Outside a given square and $\frac{1}{2}$ in. from it.

   (c) Outside of a given equilateral triangle and $\frac{1}{2}$ in. from it.

5. An isosceles triangle is to be drawn on a given fixed line segment as base. What is the locus of its vertex?

6. Find the locus of a point inside and $\frac{1}{2}$ in. from the boundary:

   (a) Of a square whose side is 2 in.

   (b) Of an equilateral triangle whose side is 3 in.

   (c) Of a circle whose radius is 1 in.

   (d) Compare these loci with those obtained in Ex. 4.

7. What is the locus of the center of a circle that has a radius of one inch, and:

   (a) Touches a given line?

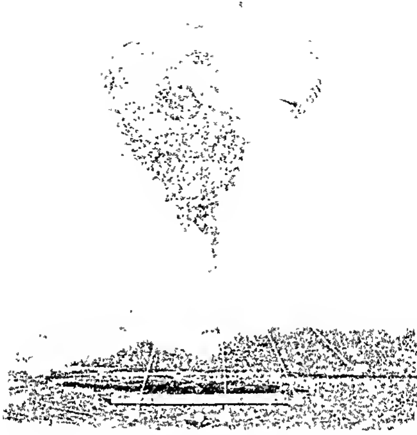
   (b) Passes through a given point?

   (c) Touches externally a given circle whose radius is 2 in.?

   (d) Touches internally a given circle whose radius is 3 in.?

8. What is the locus of the center of a circle that touches two parallel lines?

9. What is the locus of the middle point of the radius of a given circle?



THE LOCUS OF A GOLF CLUB

This is a multiple-flash photograph of Bobby Jones swinging a golf club. At very short intervals a large number of exposures were taken on the same film. In fact, the interval of time between pictures is 1/100 of a second and the exposure of each picture is 1/100,000 of a second. So the picture shows the path along which the club traveled during the complete swing, that is, the locus of the golf club. You may also be able to see the locus of Bobby's arms as he swings the club.

Courtesy of A. G. Spalding & Bros., Inc.

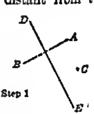
21. Point P is outside $\odot O$. What is the locus of the middle point of the chord cut off by $\odot O$ on a secant through point P?

22. The base BC of $\triangle ABC$ is fixed in position and length. What is the locus of point A if: (a) $\angle A = 90°$; (b) $\angle A = 60°$; (c) $\angle A = 120°$?

23. The base of a parallelogram is fixed in position and length, and the adjacent side has a given length. What is the locus of the intersection of its diagonals?

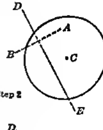
191. Method of attack. Loci are used to find a point that fulfils two given conditions. Construct the locus for each condition separately, without considering the other. Then the intersections of the two loci determine the points that fulfil both conditions.

Illustration. Find all points equally distant from two points A and B, and $\frac{1}{2}$ in. from a third point C.



Step 1

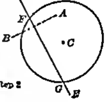
STEP 1. Disregarding point C, the locus of a point equally distant from A and B is the perpendicular bisector DE of segment AB.



Step 2

STEP 2. Disregarding A and B, the locus of a point $\frac{1}{2}$ in. from C is a circle with C as center and a radius of $\frac{1}{2}$ in.

STEP 3. Combining the two loci, the intersections F and G are each equally distant from A and B because on the first locus, and are $\frac{1}{2}$ in. from C because on the second locus.



Step 2

STEP 4. Discussion. There are 2 solutions if the distance from C to DE is less than $\frac{1}{2}$ in., 1 solution if it equals $\frac{1}{2}$ in., and no solution if it is more than $\frac{1}{2}$ in. There are never more than two solutions because a circle and a straight line can intersect in no more than 2 points.

Class Exercises

1. Find a point equally distant from three given points.

2. Find all points 2 in. from each of the given points A and B. How many solutions are there if:

   (a) A and B are 3 in. apart?
   (b) A and B are 4 in. apart?
   (c) A and B are 5 in. apart?

3. On a given circle, find a point equally distant from:

   (a) Two given parallel lines.
   (b) Two given intersecting lines.

4. Find a point equally distant from two intersecting lines and at a given distance from a given point.

5. Find a point 2 in. from a given line and 4 in. from a point which is 1 in. from the line.

6. Find a point 2 in. from a given line and 1 in. from a circle whose radius is 5 in. and whose center is 2 in. from the given line.

7. Find all points equally distant from three given lines. Discuss the number of solutions, if:

   (a) The lines intersect in 3 points.
   (b) Two of the lines are parallel and the third is a transversal.
   (c) All three lines intersect in one point.
   (d) All three lines are parallel.

8. Find a point equally distant from 2 given points and:

   (a) Equally distant from two parallel lines.
   (b) One-half inch from a given circle.

9. Find all points equally distant from 2 intersecting lines and:

   (a) One inch from their intersection.
   (b) One inch from a point on one of the lines.
   (c) Equally distant from two parallel lines.

10. Find a point equally distant from:

    (a) Two intersecting lines, and on a given circle that cuts the lines.

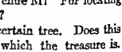
Problem. Can you tell exactly where my home is, if I tell you that it is on the north side of Tenth Street? If I tell you it is on the east side of Avenue M? If I tell you it is both on the north side of Tenth Street and the east side of Avenue M? For locating a point exactly, how many loci are needed?

A pirate buried treasure 100 ft. from a certain tree. Does this fact locate it exactly? Draw the locus on which the treasure is. If you also discover that the treasure is directly east of the tree, on what other locus can you now place it? Draw this locus. Can you now definitely locate the treasure?

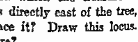
Important loci.

1. The locus of a point at a *given distance* from:

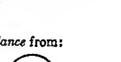
(a) A given point is a circle having the point as center and the given distance as radius.



(b) A given line is the two lines parallel to the given line and at the given distance from it.

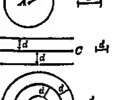


(c) A given circle is two circles whose center is the center of the given circle, and whose radii differ from that of the given circle by the given distance.



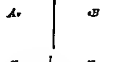
2. The locus of a point *equally distant* from:

(a) Two given points is the perpendicular bisector of the segment joining the two points.



(b) Two intersecting lines is the two lines bisecting their angles.

(c) Two parallel lines is the line parallel to the two lines and midway between them.



(e) Tangent to two parallel lines, and passing through a given point unequally distant from the lines.

(f) Tangent to two lines, and touching one of them at a given point.

(g) Tangent to a line, and to a given circle at a given point.

15. Find all points equally distant from two circles having the same center, and at a given distance from a third circle if:

(a) The third circle intersects the other two.

(b) The third circle is entirely outside the other two.

(c) The third circle has the common center of the other two. What can you say about the possibility of this case?

(d) The third circle moves from the position in (c) until it is a great distance from the other two circles.

Honor Work

16. Find the center of a circle whose radius is $\frac{1}{2}$ in., if it is tangent to a given straight line and to a given circle. Discuss the number of solutions if the line:

(a) Passes through the center of the circle.

(b) Cuts the circle.

(c) Is tangent to the circle.

(d) Is entirely outside the circle but less than $\frac{1}{2}$ in. from it.

(e) Is more than $\frac{1}{2}$ in. from the circle.

17. Find the center of a circle whose radius is $\frac{1}{2}$ in., if it is tangent to two given circles of radii 3 in. and 4 in. respectively. Discuss the solution for the different positions of the circles.

18. Find the center of a circle which touches a given straight line at a given point and touches a given circle.

19. Find the center of a circle which touches two intersecting lines and passes through a point on their angle bisector.

Applied Problems

20. Two enemy forts are at A and B, 20 mi. apart. A column of advancing troops must pass between the forts, but desires to keep as far as possible from each fort while doing so. Construct the line of march.

(b) Two intersecting lines, and also equally distant from two given points.

(c) Two given points, and a given distance from a given line.

(d) Two intersecting lines, and also equally distant from two other intersecting lines.

11. Construct a circle having a given radius and tangent to:

(a) A given line, and passing through a given point.

(b) Two given intersecting lines.

(c) A given circle, and to a given line outside the circle.

(d) A given circle, and a given line that intersects the circle.

(e) Two given circles external to each other.

(f) Two given intersecting circles.

(g) A given circle, and passing through a given point.

12. Show how to find the center of a given circle.

Optional Exercises

13. Find all points equally distant from two given lines and a given distance from a third line. Discuss the solutions if:

(a) The lines intersect in three points.

(b) The first two lines are parallel and the third is a transversal.

(c) The first two lines intersect and the third is parallel to one of them.

(d) All three lines intersect in one point.

(e) All three lines are parallel.

14. Construct a circle:

(a) Having its center on a given line, and passing through two points on the same side of the line, but unequally distant from it.

Through a given point, and tangent to a given line at another given point.

Tangent to two sides of a triangle, and having its center on the third side.

Through a given point, and tangent to a given circle at another given point.

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PROBLEMS

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14. Construct a circle:

(a) Having its center on a given line, and passing through two points on the same side of the line, but unequally distant from it.

(b) Through a given point, and tangent to a given line at another given point.

(c) Tangent to two sides of a triangle, and having its center on the third side.

(d) Through a given point, and tangent to a given circle at another given point.

(e) Tangent to two parallel lines, and passing through a given point unequally distant from the lines.

(f) Tangent to two lines, and touching one of them at a given point.

(g) Tangent to a line, and to a given circle at a given point.

15. Find all points equally distant from two circles having the same center, and at a given distance from a third circle if:

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(a) Passes through the center of the circle.

(b) Cuts the circle.

(c) Is tangent to the circle.

(d) Is entirely outside the circle but less than $\frac{1}{2}$ in. from it.

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17. Find the center of a circle whose radius is $\frac{1}{4}$ in., if it is tangent to two given circles of radii 3 in. and 4 in. respectively. Discuss the solution for the different positions of the circles.

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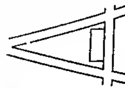
19. Find the center of a circle which touches two intersecting lines and passes through a point on their angle bisector.

Applied Problems

20. Two enemy forts are at A and B, 20 mi. apart. A column of advancing troops must pass between the forts, but desires to keep as far as possible from each fort while doing so. Construct the line of march.

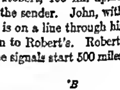
21. In a vacant lot between two streets which meet at an acute angle, some boys wish to lay out a baseball diamond, so that home plate will be equally distant from the two streets and 90 yds. from a large building which extends between the streets. Show how to locate the plate on a plan of the grounds.



22. A large tree is 10 ft. from a straight fence. A man, dying, tells his son that he buried treasure 20 ft. from the fence and 60 ft. from the tree. Locate all the points where the son should dig.

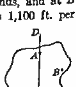
23. Two amateur radio fans, John and Robert, 100 mi. apart, hear some new signals and desire to locate the sender. John, with a loop aerial, finds that the sending station is on a line through his home making a 45° angle with the direction to Robert's. Robert, who has an outside anteana, decides that the signals start 500 miles from his station. Construct a figure showing the probable location of the new station.

24. It is desired to locate a railway station on railroad CD, so that it will be equally distant from two villages A and B, which are unequally distant from the railroad. Construct a plan and locate the station on it.



25. An explosion is heard at two towns A and B, which are 25,000 ft. apart. At A the sound arrives 20 seconds, and at B 16 seconds, after the flash is seen. Sound travels 1,100 ft. per second. Make a drawing and construct the possible locations of the explosion.

26. When a physicist wishes to find the center of gravity of an irregular object, he suspends it from point A and also suspends a plumb line from the same point. The plumb line will pass through the center of gravity. Show that, if he now suspends it from another point B, in the same way, he can find the center of gravity.



SPACE GEOMETRY (*Optional*)

1. What is the locus of a point in your classroom:
   - (*a*) Three feet above the floor?
   - (*b*) Four feet from the front wall?
   - (*c*) Both 3 ft. above the floor and 4 ft. from the front wall?
   - (*d*) Two feet from the right hand wall?
   - (*e*) Three feet above the floor, 4 ft. from the front wall and 2 ft. from the right hand wall?

2. What is the locus in space of a point:
   - (*a*) Equally distant from two given points?
   - (*b*) Three inches from a given point?
   - (*c*) One-half inch from a given plane?
   - (*d*) One inch from a given straight line?
   - (*e*) Two inches outside a given sphere?
   - (*f*) Equally distant from two given parallel planes?
   - (*g*) Equally distant from two given intersecting planes?

3. What is the locus of a straight line:
   - (*a*) Perpendicular to a given line at a given point on the line?
   - (*b*) Parallel to a given plane and passing through an outside point?
   - (*c*) Intersecting two given parallel lines?

4. What is the locus of the center of a sphere:
   - (*a*) Passing through two given points?
   - (*b*) Touching two intersecting planes?
   - (*c*) Touching a plane at a given point?
   - (*d*) Touching a given sphere at a given point?

5. What is the locus of the center of a sphere having a given radius, and:
   - (*a*) Passing through a given point?
   - (*b*) Touching a given plane?
   - (*c*) Touching a given line?
   - (*d*) Touching a given sphere externally?
   - (*e*) Touching a given plane along a given line in the plane?
   - (*f*) Touching a given sphere along a given circle on the sphere?

Angles measured by arcs.

1. A central angle is measured by its arc (§ 162).

2. An inscribed angle is measured by one-half its arc (§ 165).

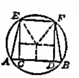
3. An angle formed by a tangent and a chord is measured by one-half its arc (§ 170).

4. An angle, inside the circle, formed by two chords is measured by one-half the sum of the opposite intercepted arcs (§ 171).

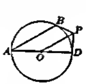
5. An angle inscribed in a semicircle is a right angle (§ 166).

Review Exercises: Honor Work

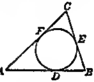
1. If equal distances AC and BD are taken on chord AB, and perpendiculars EC and FD are erected to meet the circle at E and F, then $ABFE$ is an isosceles trapezoid.



Ex. 1.



Exs. 2, 3.



Ex. 4.

2. If BP and PD are tangents to $\odot O$, and AD is a diameter, prove that AB is parallel to OP.

3. If AOD is a diameter, BP a tangent, and OP parallel to AB, prove that PD is a tangent.

4. The sides of a circumscribed triangle are 42, 50, and 56. Find the lengths from the vertices to the points of tangency.

(Let $AD = x$, $DB = y$, then $CF = AC - x$, $CE = CB - y$.)

5. A square circumscribed about a circle is double the square inscribed in the same circle.

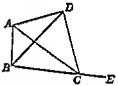
6. Quadrilateral $ABCD$ is circumscribed about $\odot O$. Prove that the sum of $\angle AOB$ and $\angle COD$ is a straight angle.

7. The side of an equilateral triangle circumscribed about a circle is twice the side of an equilateral triangle inscribed in the same circle.

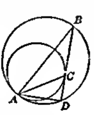
8. If the sum of two opposite angles of a quadrilateral is two right angles, the quadrilateral can be inscribed in a circle.

9. If exterior $\angle DCE$ equals $\angle BAD$, show that $\angle BAC = \angle BDC$. (Show that $ABCD$ can be inscribed in a circle.)

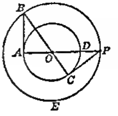
10. In quadrilateral $ABCD$, if $\angle DAC = \angle DBC$, prove that $\angle DAB$ and $\angle DCB$ are supplementary.



Exs. 9, 10.



Ex. 11.

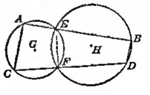


Ex. 13.

11. Two circles are tangent internally at A, and chord BD of the larger is tangent to the smaller at C. Prove that CA bisects $\angle BAD$.

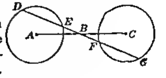
12. The perpendicular bisector of one of two parallel chords bisects the other.

13. To construct a tangent to $\odot ACD$ from point P, draw $\odot PBE$ with O as center and OP as radius. Then construct $AB \perp AP$, and line BC through O. Prove that PC is the required tangent.



14. Construct a circle tangent to three equal circles.

15. If AB and CD are drawn through the points of intersection, E and F, of ⊙ G and H, then $AC \parallel BD$.

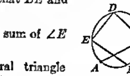


16. If DAE is a tangent through vertex A of inscribed $\triangle ABC$, prove that the three angles at A equal respectively the three angles of the triangle.

17. If $\odot A = \odot C$ and $AB = BC$, prove that $DE = FG$.

18. Inscribe a circle whose radius is 1 in. in a semicircle whose radius is 3 in.

19. In $\triangle ABC$, the circle on BC as a diameter cuts AB and AC in D and E respectively. Prove that BE and CD are altitudes of $\triangle ABC$.



20. If arc $\widehat{AB}$ equals $60°$, find the sum of $\angle E$ and $\angle C$.

21. The altitude of an equilateral triangle is three times the radius of its inscribed circle.

22. A circle is divided into five arcs, so that each arc, beginning with the second, is $10°$ larger than that just preceding it. Find all the angles of the inscribed polygon and the angles formed by the diagonals.

23. Right triangles inscribed in equal circles are congruent, if a leg of one equals a leg of the other.

24. Right triangles inscribed in equal circles are congruent, if an acute angle of one equals an acute angle of the other.

25. Find the locus of the middle point of a chord which passes through a fixed point on the circle.

26. Find the locus of the middle point of a chord which passes through a fixed point inside the circle.

27. Construct a triangle, given two sides and the sum of the angles opposite them.

28. Construct an isosceles triangle, given a base angle and the perimeter.

29. Construct four equal circles in a square so that each circle will touch two sides of the square and two other circles.

30. Within a given circle, construct three equal circles, each of which will be tangent to all three of the others.

31. The difference of any two sides of a triangle equals the difference of the segments of the third side made by the point of contact of the inscribed circle.

32. If two adjacent sides of an inscribed quadrilateral are equal, perpendiculars from the vertex of their included angle on the other two sides, produced if [illegible] ual.

BOOK THREE

PROPORTION; SIMILAR FIGURES

196. A **ratio** is the quotient obtained when one quantity is divided by another of the same kind.

A ratio may be expressed as a common fraction, for example, the ratio of 30 to 75 is $\frac{30}{75}$ or $\frac{2}{5}$.

197. A **proportion** is a statement that two ratios are equal.

A proportion is written $a : b = c : d$, or $\frac{a}{b} = \frac{c}{d}$, and is read, "$a$ is to b as c is to d." The *terms* of the proportion are the four quantities a, b, c, and d. The first and fourth terms, a and d, are called the *extremes*, and the second and third terms, b and c, are called the *means*.

Although it is necessary that the terms of a ratio be expressed in the same unit, it is not necessary that the four terms of a proportion be of the same kind. For example, if an iron rod 6 in. long weighs 2 lbs., and one 9 in. long weighs 3 lbs., we cannot express as a ratio 6 in. : 2 lbs., but we can say that $\frac{6 \text{ in.}}{9 \text{ in.}} = \frac{2 \text{ lbs.}}{3 \text{ lbs.}}$. In fact, this is the most common use of proportion throughout science. Similarly, the terms of one ratio of a proportion may be expressed in inches and of the other in feet or miles. The ratio itself is not expressed in terms of any unit but is an abstract number. For example, the ratio $\frac{6 \text{ in.}}{9 \text{ in.}}$ is $\frac{2}{3}$, not $\frac{2}{3}$ in.

198. A **fourth proportional** is the fourth term, d, in the proportion.

199. If the two means b and c are equal to each other, either is called a mean **proportional.**

In the proportion $\frac{a}{b}=\frac{b}{d}$, b is the mean proportional.

Theorems on Proportion

200. *In a proportion, the product of the extremes equals the product of the means.*

$\frac{a}{b}=\frac{c}{d}$ by hypothesis. Multiplying by the common denominator, bd, $ad=bc$ is obtained (Ax. 5).

201. *The mean proportional between two numbers is the square root of their product.*

$\frac{a}{b}=\frac{b}{c}$ by hypothesis. Then $b^2=ac$ (200), and $b=\sqrt{ac}$.

202. *If three terms of one proportion equal respectively the three corresponding terms of another proportion, their remaining terms are equal.*

$\frac{a}{b}=\frac{c}{x}$ and $\frac{a}{b}=\frac{c}{y}$ by hypothesis. Then $ax=bc$ and $ay=bc$ (§ 200). $ax=ay$ (Ax. 1), and $x=y$ (Ax. 6).

203. *If the product of two quantities equals the product of two others, one pair may be made the means and the other pair the extremes of a proportion.*

$km=pq$ by hyp. Dividing by mp, $\frac{km}{mp}=\frac{pq}{mp}$, or $\frac{k}{p}=\frac{q}{m}$ (Ax. 6).

204. *In any proportion, the terms are also in proportion by addition.*

$\frac{a}{b}=\frac{c}{d}$ by hypothesis. Adding 1, $\frac{a}{b}+1=\frac{c}{d}+1$, or $\frac{a+b}{b}=\frac{c+d}{d}$ (Ax. 3). Also $\frac{a+b}{a}=\frac{c+d}{c}$.

Class Exercises

1. Find a fourth proportional to 2, 3, and 4; to 5, 8, and 11; to 2.4, 6, and 8.3.

2. Find the mean proportional between 3 and 12; between 2 and 4; between 1.8 and 3.2.

3. If $x=\sqrt{ab}$, write a proportion in terms of x, a, and b.

4. If $x=\sqrt{12ab}$, write a proportion beginning with $3a$.

5. If $x=\sqrt{3}$, write a proportion in terms of x and numbers.

6. If $ef=gh$, write eight proportions in terms of these letters.

7. If $k^2=mn$, write three proportions in terms of these letters

8. Determine whether the following proportions are true:

(a) $\frac{3}{5}=\frac{8}{13}$ (b) $\frac{7}{11}=\frac{14}{22}$ (c) $\frac{4.2}{5}=\frac{6.3}{7.5}$

9. Write the proportions in exercise 8 by addition.

10. Find the value of x in the following proportions:

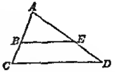
(a) $\frac{3}{x}=\frac{5}{8}$ (d) $\frac{3}{x}=\frac{4}{7}$ (g) $\frac{5}{x}=\frac{x}{8}$

(b) $\frac{2}{4}=\frac{5}{x}$ (e) $\frac{x+2}{4}=\frac{5}{8}$ (h) $\frac{3.2}{x}=\frac{x}{12.8}$

(c) $\frac{3}{5}=\frac{8}{x}$ (f) $\frac{2}{x}=\frac{x}{32}$

11. Find $\frac{x}{y}$, if $2x=3y$; if $5x=y$.

12. If $\frac{y}{3}=\frac{x}{y}$ and $\frac{y}{3}=\frac{x}{6}$, find x and y.

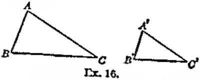


13. If $AB=8$, $BC=4$, $AE=10$, and $ED=5$, are the lines AC and AD divided into proportional segments at B and E?

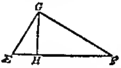
14. If $AB=6$, $BC=4$, $AE=8$, and the lines AC and AD are divided into proportional segments at B and E, find ED.

15. Write $\frac{AB}{BC}=\frac{AE}{ED}$ by addition, and substitute single lines for the sums of lines.

16. If the sides of $\triangle ABC$ are proportional respectively to the sides of $\triangle A'B'C'$, and $AB=6$, $AC=8$, $BC=9$, and $B'C'=6$, find $A'C'$ and $A'B'$.



Ex. 16.



Exs. 17, 18.

17. If $EH=3$, $HF=12$, and GH is the mean proportional between EH and HF, find GH.

18. If $EH=3$, $HF=9$, and EG is the mean proportional between EF and EH, find EG.

Geometric Reasoning Applied to Life Situations

19. In his will Jones left his wife \$36,000 and his son \$27,000, but when he died, it was found that his estate amounted to only \$56,000. How much should the court give each heir?

20. I pay \$200 a yr. taxes on property worth \$7000. What is Smith's property worth if he pays \$240 a yr. taxes?

21. William and Patrick did a job for \$90. Patrick worked 11 days and William 7 days. How should they divide the money?

22. A mining engineer finds that an ore contains lead and waste material in the ratio 3 : 8. How much lead will he get from 3850 lbs. of the ore?

23. To make fruit ice cream, use 2 c. of fruit juice, 2 c. of sugar, and 4 c. of cream. How many pounds of each will be needed to make 4 lbs. of ice cream?

205. A quantity is measured by comparing it with some accepted unit. When we say that a line is eight feet long, we mean that it is eight times as long as a unit of length

called the foot. The number, eight, is the numerical measure of the line, when the foot is the unit.

To find the number of times one line is contained in another, it is necessary to divide one by the other. If line AB is 238 in. long and line CD is 42 in. long, the ratio of AB to CD is obtained by dividing 238 by 42, and is written $\frac{238}{42}$.

We have already measured angles and arcs, using a unit called the degree, in each case, as the measure.

Two quantities of the same kind may or may not have a common unit which is contained in each a whole number of times. The lengths 8 yds. 2 ft., and 6 yds. 1 ft. have a common unit, the foot, which is contained in them 26 and 19 times respectively. But the lengths 5 in. and $\sqrt{3}$ in. have no such common unit. Any unit which is contained in one of them a whole number of times is not contained a whole number of times in the other.

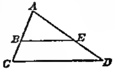
206. Two quantities of the same kind are **commensurable** if there is a common unit of measure which is contained in each of them a whole number of times. They are **incommensurable** if no such common unit of measure exists.

In this book, we prove propositions for the commensurable case only, and assume that they also hold when the quantities are incommensurable. The latter can be proved from the former by a method known as the method of limits.

207. *The ratio of two geometric quantities of the same kind is the ratio of their numerical measures.*

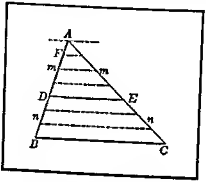
Similarly, the product of two geometric quantities, as two lines, is the product of their numerical measures.

Investigation Problem. In the figure, $BE \parallel CD$. Measure AB and BC and find their ratio. Similarly find the ratio of AE to ED. State your conclusion as a proposition. If $AB=2BC$, does $AE=2ED$? Draw a line parallel to CD bisecting AB. Compare AE and ED, if $AB=3BC$; if $AB=2\frac{1}{3}BC$.



Proposition 1

208. *A line parallel to one side of a triangle divides the other two sides into proportional segments.*



Given: [$\triangle ABC$]; $DE \parallel BC$.

To prove: $\dfrac{AD}{DB}=\dfrac{AE}{EC}$.

Proof:

Statements	Reasons
1. Suppose a common unit AF is contained in AD m times, and in DB n times. Then $\dfrac{AD}{DB}=\dfrac{m}{n}$	1. § 207.
2. Through the points of division, draw lines parallel to BC	2. § 77.
3. AE is divided into m parts, and EC into n parts, all equal.	3. § 114.
4. $\dfrac{AE}{EC}=\dfrac{m}{n}$.	4. § 207.
5. $\dfrac{AD}{DB}=\dfrac{AE}{EC}$.	5. Ax. 2.

209. Two lines are divided proportionally, when one line is to one of its segments as the other line is to its corresponding segment.

210. Corollary 1. *A line parallel to one side of a triangle divides the other two sides proportionally.*

$$\frac{AD}{DB}=\frac{AE}{EC} \text{ (§ 208)},$$

$$\frac{AD+DB}{AD}=\frac{AE+EC}{AE} \text{ (§ 204)},$$

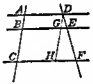
$$\frac{AB}{AD}=\frac{AC}{AE}$$

211. Corollary 2. *Parallel lines cut off proportional segments on two transversals.*

Draw $DH \parallel AC$.

Then $\frac{DG}{GH}=\frac{DE}{EF}$ (§ 208),

$$\frac{AB}{BC}=\frac{DE}{EF}$$

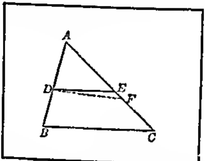


Investigation Problem. State the converse of Proposition 1. Do you think that it is true? Try to draw a triangle in which two sides are divided proportionally by a line which is not parallel to the third side. What is your conclusion?

If DH were drawn parallel to BC (in the figure for Proposition 1), and if it cuts AC at H, what must you prove about DH and DE? What ratio does $\frac{AC}{AH}$ equal? Why? What ratio does $\frac{AC}{AE}$ equal? Can you compare the length of AH with that of AE? How many terms of one of your proportions equal corresponding terms of your other proportion? Why does DH coincide with DE?

Proposition 2

* **212.** *A line that divides two sides of a triangle proportionally is parallel to the third side.*



Given: [$\triangle ABC$, D on AB, E on AC]; $\dfrac{AB}{AD}=\dfrac{AC}{AE}$.

To prove: $DE \parallel BC$.

Proof:

Statements	Reasons
1. Construct $DF \parallel BC$.	1. § 77.
2. $\dfrac{AB}{AD}=\dfrac{AC}{AE}$.	2. Hyp.
3. $\dfrac{AB}{AD}=\dfrac{AC}{AF}$.	3. § 210.
4. $AE=AF$.	4. § 202.
5. Point F is on point E.	5. $AE=AF$.
6. DF coincides with DE.	6. § 4.
7. $DE \parallel BC$.	7. DE coincides with DF which is parallel to BC.

Exercises

In the figure for Proposition 1.

1. If $AD=10$, $DB=7$, $AE=12$, and $EC=8$, is $DE \parallel BC$? Why?

2. If $DE \parallel BC$, $AD=7$, $DB=9$, and $AE=11$, find EC.

3. If $DE \parallel BC$, $AD=8$, $DB=4$, and $AC=15$, find AE.

4. $DE \parallel BC$, $AB=7$ in., and the unit of measure is 1 in. How long is the unit of measure of AE and EC, if $AC=14$ in.? If $AC=21$ cm.?

5. If $DE \parallel BC$, $AD=DB$, and $AC=17$, find AE.

6. If $AD=AE$, $DB=9$, and $DE \parallel BC$, find EC.

7. If $DE \parallel BC$, $AD=a$, $DB=b$, and $AE=c$, find EC.

8. Prove that Section **210** is still true, if DE meets BA and CA produced through A.

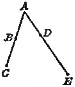
9. Prove Section 211 by drawing DC instead of DH.

10. If the ratio $\frac{AD}{DB}$ changes, DE remaining parallel to BC, must the ratio $\frac{AE}{EC}$ necessarily change? If $\frac{AD}{DB}$ increases, how does $\frac{AE}{EC}$ change?

11. If DE moves toward A but remains parallel to BC, what change takes place in the ratio $\frac{AD}{DB}$? In $\frac{AE}{EC}$? Does the proportion continue true for all positions of DE?

12. If $\frac{AD}{DB}=\frac{5}{7}$, can you tell the value of AE? Of EC? Of $\frac{AE}{EC}$?

13. AC rotates around point A but changes its length so that C travels along the fixed line BC. If $\frac{AD}{DB}=\frac{AE}{EC}$, what is the locus of point E?

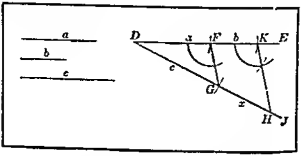


Investigation Problem. Can you find a point F on the segment DE, so that $\frac{AB}{BC}=\frac{AD}{DF}$? What method of proving lines proportional have you learned? What, then, must be the relation of the lines BD and CF? Can you construct CF?

If three line segments a, b, and c are given, construct a fourth segment x so that $\frac{a}{b}=\frac{c}{x}$.

Proposition 3

213. *A fourth proportional to three given line segments can be constructed.*



Given: Three line segments a, b, and c.

To prove: x can be constructed so that $\frac{a}{b}=\frac{c}{x}$.

Construction:

Statements	Reasons
1. On DE, take $DF=a$ and $FK=b$.	1. Post. 3.
2. On any line DJ through D, take DG equal to c.	2. Post. 3.
3. Draw FG.	3. Post. 1.
4. Draw $KH \parallel FG$. Then $GH=x$.	4. § 77.

Proof:

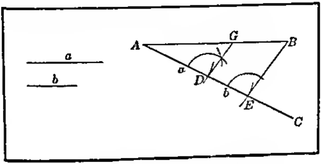
1. $KH \parallel FG$.	1. Const.
2. $\frac{a}{b}=\frac{c}{x}$.	2. § 208.

Investigation Problem. Divide a given line segment AB into three equal parts. Find three fifths of another segment CD Now divide a line segment EF into parts in the ratio of 3 to 4. If a and b are two given segments, can you now divide a segment GH into parts in the ratio of a to b? Prove your construction.

If a, b, and c are three given line segments, divide a segment KL into parts in the ratio $a : b : c$. Prove that your construction is correct.

Proposition 4

214. *A line segment can be divided into parts proportional to two given segments.*



Given: Line segments AB, a, and b.

To prove: AB can be divided into segments proportional to a and b.

Construction: Statements	Reasons
1. Draw AC to any point not on AB.	1. Post. 1
2. On AC, take $AD=a$ and $DE=b$.	2. Post. 3
3. Draw EB.	3. Post. 1.
4. Draw $DG \parallel EB$.	4. § 77.
Then AG and GB are the required segments.	
Proof:	
1. $DG \parallel EB$.	1. Const.
2. $\frac{a}{b}=\frac{AG}{GB}$.	2. § 203.

Exercises

If a, b, and c are three given line segments, construct x:

1. If $\frac{a}{b}=\frac{x}{c}$.
2. If $x=\frac{ab}{c}$.
3. If $\frac{a}{b}=\frac{b}{x}$.
4. If $x=\frac{a^2}{c}$.
5. If $x=\frac{3ab}{c}$.
6. If $x=\frac{2ab}{3c}$.

7. Construct a fourth proportional to a, b, and c; then a fourth proportional to a, c, and b. How do these two fourth proportionals compare in length?

8. Enlarge a given triangle so that a side of the enlarged triangle is to a side of the given triangle as $5:4$.

9. Divide a line segment into parts in the ratio $1:2:3$.

10. Construct x and y, if $x+y=a$ and $\frac{x}{y}=\frac{b}{c}$, where a, b, and c are given line segments.

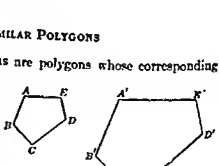
11. If C_1B_1, C_2B_2, ... are perpendicular to AB, show that the ratios

$$\frac{AB_1}{AC_1}, \frac{AB_2}{AC_2}, \frac{AB_3}{AC_3}, \ldots \text{ are equal.}$$

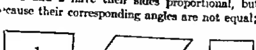
Similar Polygons

215. Similar polygons are polygons whose corresponding angles are equal and whose corresponding sides are proportional.

$ABCDE \sim A'B'C'D'E'$, if $\angle A = \angle A'$, $\angle B = \angle B'$, ..., and $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} \cdots$

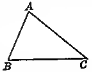


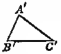
Polygons 1 and 2 have their sides proportional, but are not similar because their corresponding angles are not equal; 1 and 3



have their angles equal, but are not similar, because their corresponding sides are not proportional.

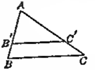
Investigation Problem. Draw a triangle having angles equal respectively to $\angle A$ and $\angle B$. Does it appear to be the same shape as $\triangle ABC$? Do you think they are similar? Try to draw a triangle having two angles equal respectively to $\angle A$ and $\angle B$ which does not appear to be similar to $\triangle ABC$. What is your conclusion? What parts of your $\triangle A'B'C'$ are known to be equal to the corresponding parts of $\triangle ABC$? How are polygons proved similar? Can you prove $\angle C'$ equal to $\angle C$? What must still be proved in order to prove $\triangle A'B'C'$ similar to $\triangle ABC$? What method of proving lines proportional have you learned? Could you place $\triangle A'B'C'$ in such a position that $B'C'$ would be parallel to BC? What sides are then proportional? How must $\triangle A'B'C'$ be placed to prove $\frac{BA}{B'A'} = \frac{BC}{B'C'}$?





Can you prove right triangles similar if you know that one acute angle of one of them equals an acute angle of the other? Does the ratio of the sides of a right triangle depend on one acute angle?

Investigation Problem. Draw a $\triangle A'B'C'$ having $\angle A'$ equal to $\angle A$, $A'B'$ one-third as long as AB, and $A'C'$ one-third as long as AC. What conclusion about $\triangle A'B'C'$ and ABC would you draw? Make another triangle in which $\angle A'$ equals $\angle A$ but the sides $A'B'$ and $A'C'$ are twice as long as AB and AC respectively. Is this triangle similar to $\triangle ABC$? If so, state your conclusion as a proposition. In order to prove $\triangle A'B'C'$ similar to $\triangle ABC$, what method seems the most reasonable? How many angles of one triangle are known to be equal to angles of the other? Could you prove these triangles similar if you knew that $\angle B'$ equalled $\angle B$? If $\triangle A'B'C'$ were placed on $\triangle ABC$, as shown in the figure, what must be known about $B'C'$ and BC in order to prove $\angle AB'C'$ equal to $\angle B$? Can you prove this relation? Will any part of the hypothesis help you? If you still cannot do these problems, look at Propositions 5 and 6.



7. Construct a fourth proportional to a, b, and c; then a fourth proportional to a, c, and b. How do these two fourth proportionals compare in length?

8. Enlarge a given triangle so that a side of the enlarged triangle is to a side of the given triangle as $5:4$.

9. Divide a line segment into parts in the ratio $1:2:3$.

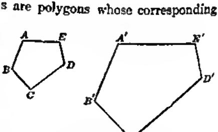
10. Construct x and y, if $x+y=a$ and $\frac{x}{y}=\frac{b}{c}$, where a, b, and c are given line segments.

11. If C_1B_1, C_2B_2, ... are perpendicular to AB, show that the ratios

$$\frac{AB_1}{AC_1}, \frac{AB_2}{AC_2}, \frac{AB_3}{AC_3}, \dots \text{ are equal.}$$

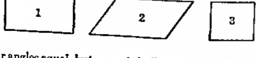
Similar Polygons

215. Similar polygons are polygons whose corresponding angles are equal and whose corresponding sides are proportional.



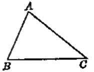
$ABCDE \sim A'B'C'D'E'$, if $\angle A = \angle A'$, $\angle B = \angle B'$, ..., and $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} \cdots$

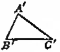
Polygons 1 and 2 have their sides proportional, but are not similar because their corresponding angles are not equal; 1 and 3



have their angles equal, but are not similar, because their corresponding sides are not proportional.

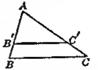
Investigation Problem. Draw a triangle having angles equal respectively to $\angle A$ and $\angle B$. Does it appear to be the same shape as $\triangle ABC$? Do you think they are similar? Try to draw a triangle having two angles equal respectively to $\angle A$ and $\angle B$ which does not appear to be similar to $\triangle ABC$. What is your conclusion? What parts of your $\triangle A'B'C'$ are known to be equal to the corresponding parts of $\triangle ABC$? How are polygons proved similar? Can you prove $\angle C'$ equal to $\angle C$? What must still be proved in order to prove $\triangle A'B'C'$ similar to $\triangle ABC$? What method of proving lines proportional have you learned? Could you place $\triangle A'B'C'$ in such a position that $B'C'$ would be parallel to BC? What sides are then proportional? How must $\triangle A'B'C'$ be placed to prove $\frac{BA}{B'A'}=\frac{BC}{B'C'}$?





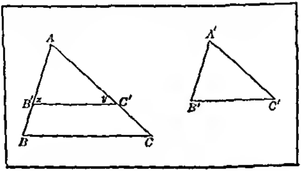
Can you prove right triangles similar if you know that one acute angle of one of them equals an acute angle of the other? Does the ratio of the sides of a right triangle depend on one acute angle?

Investigation Problem. Draw a $\triangle A'B'C'$ having $\angle A'$ equal to $\angle A$, $A'B'$ one-third as long as AB, and $A'C'$ one-third as long as AC. What conclusion about $\triangle A'B'C'$ and ABC would you draw? Make another triangle in which $\angle A'$ equals $\angle A$ but the sides $A'B'$ and $A'C'$ are twice as long as AB and AC respectively. Is this triangle similar to $\triangle ABC$? If so, state your conclusion as a proposition. In order to prove $\triangle A'B'C'$ similar to $\triangle ABC$, what method seems the most reasonable? How many angles of one triangle are known to be equal to angles of the other? Could you prove these triangles similar if you knew that $\angle B'$ equalled $\angle B$? If $\triangle A'B'C'$ were placed on $\triangle ABC$, as shown in the figure, what must be known about $B'C'$ and BC in order to prove $\angle AB'C'$ equal to $\angle B$? Can you prove this relation? Will any part of the hypothesis help you? If you still cannot do these problems, look at Propositions 5 and 6.



PROPOSITION 5

* 216. *Two triangles are similar, if two angles of one equal respectively two angles of the other.*



Given: [$\triangle ABC$ and $A'B'C'$]; $\angle A = \angle A'$ and $\angle B = \angle B'$.
To prove: $\triangle ABC \sim \triangle A'B'C'$.

Proof: STATEMENTS	REASONS
1. Place $\triangle A'B'C'$ on $\triangle ABC$, so that A' is on A, and $A'B'$ falls on AB.	1. Ax. 10.
2. Then $A'C'$ is on AC.	2. $\angle A' = \angle A$ by hyp.
3. $\angle B = \angle B' = \angle x$.	3. Hyp.
4. $B'C' \parallel BC$.	4. § 74.
5. $\angle C = \angle y = \angle C'$.	5. § 80.
6. $\frac{AB}{AB'} = \frac{AC}{AC'}$, or $\frac{AB}{A'B'} = \frac{AC}{A'C'}$.	6. § 210.
7. Similarly, by placing B' on B, $\frac{AB}{A'B'} = \frac{BC}{B'C'}$.	7. Reasons 1 to 6.
8. $\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$.	8. Ax. 2.
9. $\triangle ABC \sim \triangle A'B'C'$.	9. § 215.

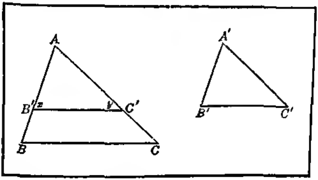
217. Corollary 1. *Right triangles are similar, if an acute angle of one equals an acute angle of the other.*

218. Corollary 2. *Triangles similar to the same triangle are similar to each other.*

219. Corollary 3. *In similar triangles, corresponding altitudes are to each other as corresponding sides.*

Proposition 6

*** 220.** ***Two triangles are similar, if an angle of one equals an angle of the other and the including sides are proportional.***



Given: [$\triangle ABC$ and $A'B'C'$]; $\angle A = \angle A'$, and $\dfrac{AB}{A'B'} = \dfrac{AC}{A'C'}$.
To prove: $\triangle ABC \sim \triangle A'B'C'$.

Proof:

Statements	Reasons
1. Place $\triangle A'B'C'$ on $\triangle ABC$, so that A' is on A, and $A'B'$ falls on AB.	1. Ax. 10.
2. Then $A'C'$ is on AC.	2. $\angle A' = \angle A$ by hyp.
3. $\dfrac{AB}{A'B'} = \dfrac{AC}{A'C'}$, or $\dfrac{AB}{AB'} = \dfrac{AC}{AC'}$.	3. Hyp.
4. $B'C' \parallel BC$.	4. § 212.
5. $\angle B = \angle x = \angle B'$.	5. § 80
6. $\angle A = \angle A'$.	6. Hyp.
7. $\triangle ABC \sim \triangle A'B'C'$.	7. § 216.

221. Corollary. *A line bisecting two sides of a triangle is parallel to the third side and equal to half the third side.*

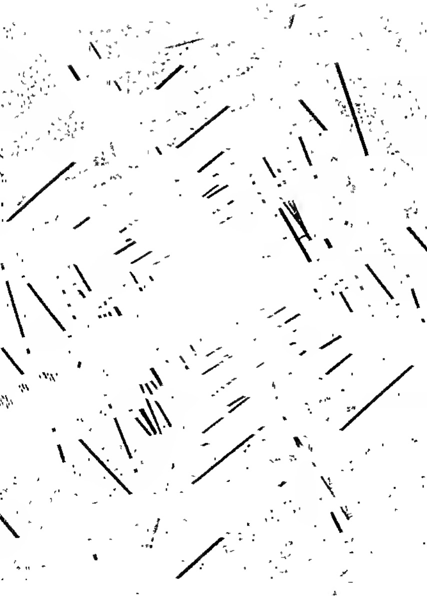
LOOKING UP INTO A BROADCASTING TOWER

This picture is a very striking illustration of the use of similar triangles. As you look up into the tower, you will notice that the triangles nearer the pointed top have shorter sides than those farther down, but have the same shape. The triangle is used in this construction because, as you have already learned, it is the most rigid polygon and gives the greatest strength.

This, however, is not the most important use of similar triangles. They are so valuable in engineering, astronomy, and surveying that much of that work would be difficult or impossible without them. Triangles can be shown to be similar by measuring angles only. And remember that angles are often more easily measured than lines. Then the sides of a triangle, extending all the way to a star, can be computed, because those sides have the same ratio as those of any smaller similar triangle that is entirely within our reach and can be measured. This makes it possible to determine long distances, or distances to inaccessible points, by measuring lines conveniently short.

Still other illustrations of similar figures are plans of buildings or machines, pictures of objects and enlargements of them, and maps of cities, states, or countries.

Photograph from Philip D. Gendreau, N. Y.

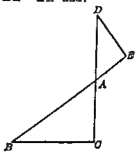


Exercises

1. Equilateral triangles are similar.

2. Are $\triangle KLM$ and PQR similar if:
   (a) $\angle K=41°$, $\angle L=73°$, $\angle Q=73°$, and $\angle R=86°$?
   (b) $\angle K=67°$, $\angle L=54°$, $\angle Q=54°$, and $\angle R=59°$?

3. Are $\triangle EFG$ and $E'F'G'$ similar if $\angle E=\angle E'$ and:
   (a) $EF=12$, $EG=15$, $E'G'=7\frac{1}{2}$, and $E'F'=6$?
   (b) $EF=9$, $EG=12$, $E'G'=9$, and $E'F'=6$?

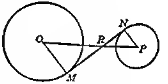


4. If $\angle C$ is a rt. $\angle$ and $DE\perp BE$, then $\triangle ADE\sim\triangle ABC$.

5. If $AB=12$, $AC=8$, $AD=9$, and $AE=6$, are $\triangle ABC$ and ADE similar? Prove your answer.

6. If $\angle B=38°$, $\angle BAC=61°$, and $\angle E=81°$, are $\triangle ABC$ and ADE similar? Prove your answer.

7. Are all isosceles triangles similar? Explain.



8. MN is the common internal tangent to ⊙ O and P. Then $\triangle OMR\sim\triangle PNR$.

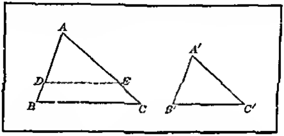
9. Make up and prove a similar exercise in which MN is a common external tangent and meets OP produced.

10. Is it possible for two triangles ABC and DEF to be similar if:
    (a) $\angle A=87°$ and $\angle E=94°$? Explain.
    (b) $\angle A=71°$, $\angle B=52°$, $\angle D=71°$, and $\angle F=85°$?

Investigation Problem. Do you think that triangles are similar if their corresponding sides are proportional? Can you place one on the other? Why not? Can you construct a new triangle on the larger having two sides equal to sides of the smaller? Are the third sides then equal? What proportions involve these third sides?

Proposition 7

* 222. *Two triangles are similar, if the three sides of one are respectively proportional to the three sides of the other*



Given: $[\triangle ABC$ and $\triangle A'B'C']$, $\dfrac{AC}{A'C'}=\dfrac{AB}{A'B'}=\dfrac{BC}{B'C'}$.

To prove: $\triangle ABC \sim \triangle A'B'C'$.

Proof:

STATEMENTS	REASONS
1. On AB, take $AD=A'B'$ and on AC, $AE=A'C'$.	1. Post. 3.
2. Draw DE.	2. Post. 1.
3. In $\triangle ABC$ and ADE, $\angle A=\angle A$.	3. Iden.
4. $\dfrac{AC}{A'C'}=\dfrac{AB}{A'B'}$.	4. Hyp.
5. $\dfrac{AC}{AE}=\dfrac{AB}{AD}$.	5. Ax. 1.
6. $\triangle ABC \sim \triangle ADE$.	6. § 220.
7. $\dfrac{AB}{AD}=\dfrac{BC}{DE}$.	7. § 215.
8. $\dfrac{AB}{A'B'}=\dfrac{BC}{B'C'}$.	8. Hyp.
9. In $\triangle ADE$ and $A'B'C'$, $DE=B'C'$.	9. § 202.
10. $AD=A'B'$ and $AE=A'C'$.	10. Const.
11. $\triangle ADE \cong \triangle A'B'C'$.	11. § 57.
12. $\triangle ABC \sim \triangle A'B'C'$.	12. Subst.

CLASS EXERCISES

1. In $\triangle ABC$ and $\triangle A'B'C'$, $AB=9$, $AC=12$, $BC=16$, $A'B'=6$, $A'C'=8$, and $B'C'=12$. Are the triangles similar? Why?

2. $\triangle ABC \sim \triangle A'B'C'$. $AB=10$, $BC=8$, $AC=9$, $A'B'=15$; find $B'C'$ and $A'C'$.

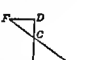
3. The sides of a polygon are 4, 6, 5, 9, and 2. The side of a similar polygon corresponding to 4 is 12. Find its other sides.

4. A tower casts a shadow 160 ft. long when a vertical 12-ft. pole casts a shadow 8 ft. long. How high is the tower?

5. A man 6 ft. in height, standing 15 ft. from a lamppost, observes that his shadow cast by the light is 5 ft. in length. How high is the light?

6. The sides of a triangle are 8, 7, and 9; if the side of a similar triangle corresponding to 7 is 10, find the other two sides.

7. If $AB \perp BD$ and $FD \perp BD$, find AB, if $BC=100$ ft., $CD=20$ ft., and $DF=50$ ft.

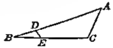


8. The base of a triangle is 20 ft.; the other sides are 16 ft. and 10 ft. A line parallel to the base cuts off 2 ft. from the lower end of the shorter side. Find the segments of the other side, and the length of the line parallel to the base.

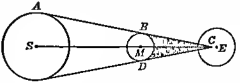
9. The sides of a triangle are 8, 10, and 12 in. respectively. If a line 9 in. long, parallel to the longest side, ends in the other two sides, find the segments into which it divides them.

10. In a trapezoid, the bases are 5 in. and 8 in., and the legs are in. and 6 in. How far must each of the legs be produced that ay m $\angle A$ a point?

Investigation P of two circles, whose and 9 in. respectively, their correspond Find how far from the the other? W rcle the line of centers ger having non internal tangent. es then $BE = \frac{1}{4}BA$, and $DE = 9$ ft., find AC.



13. There will be a total eclipse of the sun, S, whenever a part of the earth, E, is within the shadow, BCD, of the moon. If $SC=$ 93,000,000 mi., the radius of the sun is 443,000 mi., and the radius of the moon is 1,080 mi., what is the greatest possible distance of the moon, M, from the surface of the earth, at which a total eclipse is possible?



Method of attack. *To prove triangles similar*, show that two angles of one equal two angles of the other; or, that an angle of one equals an angle of the other and the including sides are proportional.

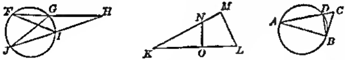
14. Two isosceles triangles are similar, if the vertex angle of one equals the vertex angle of the other.

15. Two isosceles triangles are similar, if a base angle of one equals a base angle of the other.

16. If two chords AB and CD intersect at E, $\triangle ACE \sim \triangle BDE$.

17. Prove that $\triangle FHI \sim \triangle JHG$.

18. In $\triangle KLM$, $NO \perp KL$ and M is a right angle, prove that $\triangle KNO \sim \triangle KLM$.



Ex. 17. Ex. 18. Ex. 20.

19. The altitude on the hypotenuse of a right triangle cuts the triangle into two similar triangles.

20. If AB is a diameter, and BC a tangent, prove that $\triangle ABD \sim \triangle BDC$.

21. If one of two similar triangles is isosceles, the other is also isosceles.

22. If the diagonals of a quadrilateral divide each other into proportional segments, the quadrilateral has two sides parallel.

CLASS EXERCISES

1. In $\triangle ABC$ and $\triangle A'B'C'$, $AB=9$, $AC=12$, $BC=16$, $A'B'=6$, $A'C'=8$, and $B'C'=12$. Are the triangles similar? Why?

2. $\triangle ABC \sim \triangle A'B'C'$. $AB=10$, $BC=8$, $AC=9$, $A'B'=15$; find $B'C'$ and $A'C'$.

3. The sides of a polygon are 4, 6, 5, 9, and 2. The side of a similar polygon corresponding to 4 is 12. Find its other sides.

4. A tower casts a shadow 160 ft. long when a vertical 12-ft. pole casts a shadow 8 ft. long. How high is the tower?

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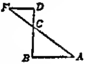
6. The sides of a triangle are 8, 7, and 9; if the side of a similar triangle corresponding to 7 is 10, find the other two sides.

7. If $AB \perp BD$ and $FD \perp BD$, find AB, if $BC=100$ ft., $CD=20$ ft., and $DF=50$ ft.



8. The base of a triangle is 20 ft.; the other sides are 16 ft. and 10 ft. A line parallel to the base cuts off 2 ft. from the lower end of the shorter side. Find the segments of the other side, and the length of the line parallel to the base.

9. The sides of a triangle are 8, 10, and 12 in. respectively. If a line 9 in. long, parallel to the longest side, ends in the other two sides, find the segments into which it divides them.

10. In a trapezoid, the bases are 5 in. and 8 in., and the legs are in and 6 in. How far must each of the legs be produced that ay $\angle A$ a point?

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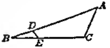


32. For each of the following proportions, read the triangles which must be proved similar, without attempting to draw a figure:

(a) $\frac{AG}{GC}=\frac{AD}{GF}$; (b) $\frac{PM}{PQ}=\frac{PR}{PS}$; (c) $\frac{AE}{A'E'}=\frac{ED}{E'D'}$; (d) $\frac{HM}{HK}=\frac{KH}{LH}$.

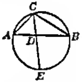
33. In the acute $\triangle ABC$, the altitudes BD and CE are drawn. Prove $\frac{BD}{CE}=\frac{AB}{AC}$.

34. If, using the same hypothesis, BD and CF intersect at F, prove that $\frac{BF}{CF}=\frac{EF}{DF}$.

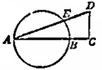
35. The diagonals of a trapezoid divide each other into proportional segments.

36. If a tangent and a secant are drawn from a point outside a circle, the tangent is a mean proportional between the secant and its external segment.

37. The bisector of $\angle C$ of $\triangle ABC$ meets AB at D, and the circumscribed circle at E. Prove $\frac{CA}{CD}=\frac{CE}{CB}$.



Ex. 37.



Exs. 39, 40.

38. In the right $\triangle ABC$, CD is the altitude on the hypotenuse. Prove that $\frac{AC}{CB}=\frac{AD}{CD}$.

39. If AB is a diameter and $CD\perp AB$ extended, then $\frac{AB}{AD}=\frac{AE}{AC}$.

40. If AB is a diameter and $\frac{AB}{AD}=\frac{AE}{AC}$, then $CD\perp AB$.

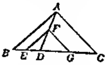
41. In similar triangles bisectors of corresponding angles have the same ratio as a pair of corresponding sides.

42. In similar triangles corresponding medians have the same ratio as a pair of corresponding sides.

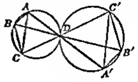
23. If two polygons, $ABCDE$ and $A'B'C'D'E'$, are similar, prove that $\triangle ABC \sim \triangle A'B'C'$.

24. If D is any point on BC, and E, F, and G bisect DB, DA, and DC respectively, then $\triangle EFG \sim \triangle ABC$.

25. If D is any point inside $\triangle ABC$, and E, F, and G bisect DB, DA, and DC respectively, then $\triangle EFG$ is similar to $\triangle ABC$.



Ex. 24



Ex. 26.

26. If two circles are tangent at D, and AA', BB', and CC' pass through D and end in the circles, then $\triangle ABC \sim \triangle A'B'C'$.

27. A line bisecting one side of a triangle and parallel to a second side bisects the third side and equals one-half the second side.

28. *A line bisecting one leg of a trapezoid and parallel to the bases bisects the other leg.*

29. *A line bisecting the legs of a trapezoid is parallel to the bases and equal to half their sum.*

30. The line bisecting two adjacent sides of a quadrilateral is equal and parallel to the line bisecting the other two sides.

31. The lines, joining in order the middle points of the four sides of any quadrilateral, form a parallelogram.

223. Method of attack. *To prove lines proportional,* show that they are corresponding sides of similar triangles.

Choose one triangle whose sides are the numerators of the proportion, and another whose sides are the denominators. If unsuccessful with this choice, make the terms of each fraction the sides of a triangle. Mark the angles of one of the triangles x, y, z. Then mark the corresponding angles of the other triangle x', y', z', making use of the fact that angles opposite corresponding sides are the corresponding angles. Look for identical angles, angles measured by equal arcs, etc.

32. For each of the *following* proportions, read the triangles which must be proved similar, without attempting to draw a figure:

(a) $\frac{AG}{GC}=\frac{AD}{GF}$; (b) $\frac{PM}{PQ}=\frac{PR}{PS}$; (c) $\frac{AE}{A'E'}=\frac{ED}{E'D'}$; (d) $\frac{HM}{HK}=\frac{KH}{LH}$.

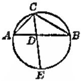
33. In the acute $\triangle ABC$, the altitudes BD and CE are drawn. Prove $\frac{BD}{CE}=\frac{AB}{AC}$.

34. If, *using the same hypothesis*, BD *and* CF *intersect at* F, prove that $\frac{BF}{CF}=\frac{EF}{DF}$.

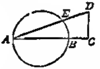
35. The diagonals of a trapezoid divide each other into proportional segments.

36. If a tangent and a secant are drawn from a point outside a circle, the tangent is a mean proportional between the secant and its external segment.

37. The bisector of $\angle C$ of $\triangle ABC$ meets AB at D, and the circumscribed circle at E. Prove $\frac{CA}{CD}=\frac{CE}{CB}$.



Ex. 37.



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39. If AB is a diameter and $CD \perp AB$ extended, then $\frac{AB}{AD}=\frac{AE}{AC}$.

40. If AB is a diameter and $\frac{AB}{AD}=\frac{AE}{AC}$, then $CD \perp AB$.

41. In similar triangles bisectors of corresponding angles have the same ratio as a pair of corresponding sides.

42. In similar triangles corresponding medians have the same ratio as a pair of corresponding sides.

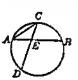
43. If the sides AB and DC of inscribed quadrilateral $ABCD$ are extended to meet at E, prove that $\frac{AE}{DE}=\frac{CE}{BE}$.

44. If C is the middle point of $\widehat{AB}$, prove that AC is a mean proportional between CD and CE.

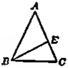
45. State and prove the converse of Ex. 44.

46. If two circles are tangent to each other at P, the chords formed by a line through P are proportional to the diameters.

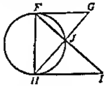
47. If two circles are tangent at P, secants through P are cut proportionally by the circles.



Ex. 44.



Ex. 48.



Ex. 49.

48. In $\triangle ABC$, $\angle B=2\angle A$, and BE bisects $\angle B$. Prove that BC is a mean proportional between CE and AC.

49. If FH is a diameter, and FG and HI are tangents, FH is a mean proportional between FG and HI.

224. Method of attack. *To prove two products equal,* make them the means and extremes respectively of a proportion, and prove the proportion. Then the product of the means equals the product of the extremes. When one of the products is a square, make it a mean proportional. Conversely, if two products are equal, form a proportion and use this proportion to help prove triangles similar.

50. If the lines AB and CD intersect at E so that $AE\times EB=CE\times ED$, prove that $\angle ADE=\angle CBE$.

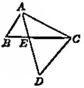
51. If two chords intersect inside a circle, the product of the segments of one chord equals the product of the segments of the other.

52. If AD bisects $\angle A$ and $AB \times AC = AD \times AE$, then $\angle B = \angle D$.

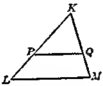
53. If two secants are drawn from a point, the product of one secant and its external segment equals the product of the other secant and its external segment.

54. In inscribed $\triangle ABC$, the product of AB and AC equals the product of the altitude AD and the diameter AE.

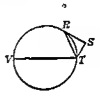
55. If, in $\triangle KLM$, PQ is parallel to LM, prove that $KL \times KQ = KP \times KM$.



Ex. 52.



Ex. 55.



Ex. 57.

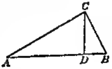
56. State and prove the converse of Ex. 55.

57. If RS is tangent to the circle, $TS \perp RS$, and VT is a diameter, then $\overline{RT}^2 = VT \times ST$.

58. If AB is the diameter of a circle, AC a chord, and $CD \perp AB$, then $\overline{AC}^2 = AD \times AB$.

59. If AB is the diameter of a circle, AC a chord, and D a point on AB such that $\overline{AC}^2 = AD \times AB$, then $CD \perp AB$.

60. The product of the hypotenuse of a right triangle and the altitude on it equals the product of the legs.



61. If CD is the altitude on the hypotenuse of right $\triangle ABC$, then $\overline{CD}^2 = AD \times DB$.

62. Using the same hypothesis, $\overline{BC}^2 = AB \times DB$.

63. Using the same hypothesis, $AC \times CD = AD \times CB$.

64. If, in $\triangle ABC$, $\overline{AC}^2 = AD \times AB$, prove that $\frac{CD}{CB} = \frac{AD}{AC}$.

65. If BD and CE are altitudes of $\triangle ABC$, then $CE \times AB = BD \times AC$.

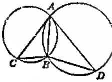
66. If one chord is bisected by another chord, the square of a

segment of the former equals the product of the segments of the latter.

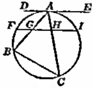
Honor Exercises

67. AC and AD are tangents to $\odot ABD$ and ABC respectively. Prove that $\overline{AB}^2 = BC \times BD$.

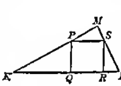
68. If FI is parallel to tangent DE, then $AB \times AG = AC \times AH$.



Ex. 67.



Exs. 68, 69.



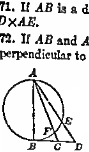
Ex. 70.

69. If $AB \times AG = AC \times AH$, then FI is parallel to tangent DE.

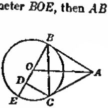
70. If $\angle M$ is a right angle and $PQRS$ is a square, then $\overline{QR}^2 = KQ \times RL$.

71. If AB is a diameter and BD is a tangent, then $AC \times AF = AD \times AE$.

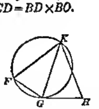
72. If AB and AC are tangents to $\odot O$ from point A, and CD is perpendicular to diameter BOE, then $AB \times CD = BD \times BO$.



Ex. 71.



Ex. 72.



Ex. 73.

73. If AD and BC are the parallel sides of trapezoid $ABCD$, and diagonals AC and BD intersect in E, prove that $AE \times EB = DE \times EC$.

74. If DB is a perpendicular to the diameter AC, from D, a point on chord AE, then $AE \times AD = AC \times AB$.

75. If GH is a tangent and GK bisects $\angle FKH$, then $\overline{GK}^2 = HK \times FF$

76. (a) In $\triangle ABC$, can $\angle A$ grow larger without any corresponding change in the sides of the triangle? Can the sides grow longer without any corresponding change in the size of $\angle A$?

(b) If the three sides of $\triangle ABC$ are given, can you construct the three angles, A, B and C? If the three angles of $\triangle ABC$ are given, can you construct the sides?

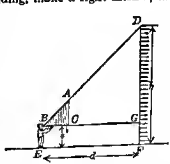
(c) Do the sides of a triangle determine the angles? Do the angles of a triangle determine the sides? Do the angles of a triangle give any information about the sides? Given this information and, in addition, any one of the sides, could you construct the triangle?

77. Determine the relations of the sides in the following triangles, two of whose angles are: (a) 60°, 60°; (b) 60°, 30°; (c) 45°, 90°.

78. Construct each of the triangles in Ex. 77, if the shortest side is 2 in.

Applied Problems

To measure the height of a building, make a right $\triangle ABC$, and suspend a weight from corner A. Hold the triangle at the level of the eye so that the edge AC will be vertical (it will be vertical when the plumb line hangs parallel to the edge), and walk toward or from the building until the highest point D is in line with B and A. Let e be the height of the eye from the ground, d the distance of the observer from the building, and h the height of the building.



79. If $\triangle ABC$ is isosceles, prove that $h=d+e$.

80. If $\triangle ABC$ is isosceles, what is the height of the building, when a boy, whose eye is $4\frac{1}{2}$ ft. above the ground, finds that his distance from the building is 18 ft.?

81. If the triangle is not isosceles, but BC is represented by a and AC represented by b, prove that $h=\frac{bd}{a}+e$.

82. If $AC=5$ in., $BC=3$ in., $BE=5$ ft., and $EF=24$ ft., find DF.

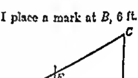
83. If the triangle has an altitude of 9 in. and a base of 3 in. and the boy, whose eye is 5 ft. from the ground, stands 20 ft. from the building, what is the height of the building?

84. A vertical yardstick casts a shadow 20 in. long at the same time that a building casts a shadow 30 ft. long. How high is the building?

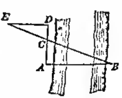
85. A man six feet tall, standing 28 ft. from an arc light, notices that his shadow is 8 ft. long. How long would it become, if he walked 6 ft. toward the light?

86. The height of a tree can be found by placing a mirror horizontally on the ground, and walking back until the top of the tree can be seen in the mirror. If Paul, whose eye is 4 ft. from the ground, finds that he sees the top of the tree in the mirror, when he is 5 ft. from the mirror and the mirror is 30 ft. from the tree, how high is the tree?

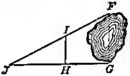
87. To find the height of a building, I place a mark at B, 6 ft. from the ground. Then I walk back a distance from the building and, holding my ruler vertically at arm's length from me, so that the lower end is in line with my eye and the base A of the building, I notice the points E and F. If E is at $1\frac{1}{2}$ in. and F at 10 in., what is the height of the building?



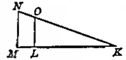
88. A surveyor wishes to find the distance AB for a bridge. He measures AC 20 ft. at right angles to AB, and sets a stake at C. Then he walks along the line CE and sets a stake at E. Next he walks along CD to a point D, where ED is at right angles to CD. If $CD=8$ ft. and $ED=22$ ft., what is the distance AB?



89. Chester found the length of a pond FG by measuring GJ at right angles to FG. He found this distance to be 40 ft., and also found that $JH = 15$ ft., and $HI = 12$ ft. at right angles to JG. What result did he get and how did he get it?

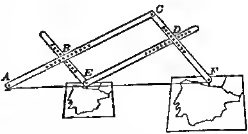


90. An inaccessible distance LK can be found by extending KL to M and taking LO and MN perpendicular to KM. If $LM = 10$ ft., $LO = 16$ ft., and $MN = 20$ ft., find LK.



91. In the camera, the light proceeds in straight lines from the object to the image, which is inverted. If the camera is 6 in. deep and is held 10 ft. from a bush 4 ft. tall, what height will the picture have? If Katherine wishes the picture of the bush to be just 2 in. in height, how far from it should she hold the camera?

92. The pantograph consists of four strips AC, ED, BE, and CF, jointed to form a $\square BEDC$. It has a point at E and a pencil at F. If A is held stationary, while E traces a figure, F will trace a similar figure. If AC is 20 in. long, and I wish to make a figure whose length is three times that of the figure at E, what lengths must I make AB and BC?

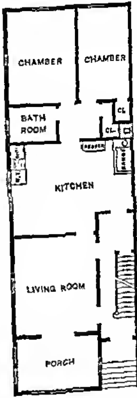


93. If, in Ex. 92, CF is 12 in., find the length of CD.

94. When a ruler is held 28 in. from the eye, the diameter of the moon seems to be about $\frac{1}{4}$ in. If the moon is 240,000 miles away, what is its real diameter?

95. The sun appears the same size as the moon. What is its diameter, if it is 93,000,000 miles away?

The plan shown in the diagram is that of a building 50 ft. long. By making measurements on the plan, answer the following questions:



96. What is the width of the building?

97. Find the dimensions of the living room and of the kitchen.

98. What is the width of the doorway between the living room and porch? What width is the hall beside the stairway?

99. Is there room enough against the wall between living room and kitchen for a piano, the length of a piano being $4\frac{1}{2}$ ft.?

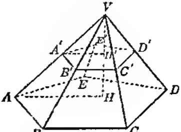
100. If a single width of carpet is to be laid the full length of the hall, how many yards will be required?

101. In cutting cloth with a pattern which was too small, John's mother cut the cloth 2 in. out from the pattern all around. Prove that, if the length and width of the pattern were not equal, the resulting piece of cloth did not have the same shape as the pattern.

102. If the length of the pattern is twice the width, how much should she increase the length if she increases the width by 4 in.?

Space Geometry (*Optional*)

1. In the pyramid $VABCDE$, the plane $A'D'$, parallel to the base AD, cuts the altitude VH at H'. Prove that:



(*a*) $\dfrac{VA'}{VA}=\dfrac{VB'}{VB}$.

(*b*) $\triangle VA'B' \sim \triangle VAB$.

(*c*) $\dfrac{VB'}{VB}=\dfrac{A'B'}{AB}$.

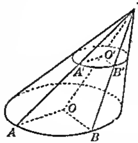
(*d*) $\dfrac{VB'}{VB}=\dfrac{B'C'}{BC}$.

(*e*) $\dfrac{A'B'}{AB}=\dfrac{B'C'}{BC}$. (*f*) $\dfrac{VH'}{VH}=\dfrac{VA'}{VA}$.

(*g*) Polygon $A'B'C'D'E' \sim$ polygon $ABCDE$.

2. The figure below is a circular cone. Its base is a circle, and its lateral surface could be traced by VA if A moved around on the circle while V stayed fixed. O is the center of the circle and VO is the axis of the cone.

If the plane $O'A'B'$ is parallel to plane OAB, prove that:



(*a*) $\dfrac{VA'}{VA}=\dfrac{VO'}{VO}$. (*c*) $\dfrac{VO'}{VO}=\dfrac{B'O'}{BO}$.

(*b*) $\dfrac{VO'}{VO}=\dfrac{A'O'}{AO}$. (*d*) $\dfrac{A'O'}{AO}=\dfrac{B'O'}{BO}$.

(*e*) Is the intersection a circle? If A' and B' are any two points on the intersection, show that $A'O'=B'O'$.

3. Draw a figure to represent the way you think the lateral surface of a paper pyramid would look if it were cut along an edge and spread out on the plane. In the same way represent the cone.

A Self-Measuring Test

1. What is the method of proving the product of two line segments equal to the product of two other line segments?

2. State two methods of proving lines proportional.

3. Give three methods of proving triangles similar.

4. Give four methods of proving triangles congruent.

5. Are similar triangles necessarily congruent? Are congruent triangles necessarily similar?

6. For what purposes do we prove triangles similar?

7. For what purposes do we prove triangles congruent?

8. If two triangles have the three sides of one equal respectively to the three sides of the other, are they similar? Are they congruent?

9. If two triangles have the three angles of one equal respectively to the three angles of the other, are they similar? Are they congruent?

10. Describe a method of finding a distance without measuring it directly.

11. Define *similar triangles.*

12. Why do we not prove that a line parallel to one side of a triangle divides the other sides proportionally, by using similar triangles?

13. On what proposition in Book One does the proof of the proposition mentioned in Ex. 12 depend?

14. Besides the methods for proving any two triangles similar, what extra theorem have you learned for proving right triangles similar? For proving right triangles congruent?

15. State four propositions about the measurement of angles by arcs.

16. In equal circles, are central angles proportional to their arcs?

17. In equal circles, are inscribed angles proportional to their arcs?

18. After proving triangles similar, how can you select the corresponding sides?

19. State a proposition in proportion in which two quantities are proved equal.

20. What test can you apply to determine whether two numerical ratios are equal?

21. If a central angle and an angle formed by a tangent and a chord intercept the same arc, what is the ratio of the angles?

22. Define a *chord*. State two propositions about chords.

23. Define *tangent*. State two propositions about tangents.

24. To prove that similar triangles are congruent, what extra fact must be known?

25. Define *mean proportional; fourth proportional; extremes; means.*

26. State a fact about the altitudes of similar triangles which you have learned.

Geometric reasoning applied to life situations (Optional)

Assume that whenever wages are raised, prices are raised too. Does it necessarily follow that: (*a*) If wages are not raised, prices are not raised? (*b*) If prices are not raised, wages are not raised? (*c*) If wages are raised, prices are raised? (*d*) If prices are raised, wages are raised?

Completion Test (*10 min.*)

After the number of the question, write the omitted word.

1. If any angle of one isosceles triangle equals the corresponding angle of another isosceles triangle, then the two triangles are . . . .
2. A line parallel to one side of a triangle cuts a second side into segments 4 and 5. The third side is 12. The shorter segment of the third side is . . . .
3. The line segment joining the middle points of two sides of a triangle equals . . . the third side.
4. Corresponding sides of two similar triangles are in the ratio 1 : 4. The perimeters of these triangles are in the ratio . . . .
5. If a building casts a shadow 15 ft. long at the same time that a yardstick held vertically casts a shadow 1 ft. long, the building is . . . ft. high.

NUMERICAL TEST (*10 min.*)

1. On a map, Wyoming is represented by a rectangle 7 in. long and $5\frac{1}{2}$ in. wide. If the length of Wyoming is 350 mi., find its width.

2. The sides of a triangle are 6, 8 and 9. A line segment whose length is 3, parallel to the longest side, ends in the other two sides. Find the shorter segment of side 8.

3. The base and altitude of a triangle are 6 and 4 respectively. If the base of a similar triangle is 9, find the corresponding altitude.

4. A monument on level ground casts a shadow 150 ft. long while a pole 12 ft. high casts a shadow 10 ft. long. How high is the monument?

5. The sides of a triangle are 2, 3 and 4. Find the sides of a similar triangle whose perimeter is 45.

MATCHING TEST (*10 min.*)

Match numbers and letters so as to make a correct statement.

1. A ratio
2. A proportion
3. Locus
4. We prove products equal
5. Similar polygons
6. Mean proportional
7. Commensurable
8. We prove lines proportional
9. Fourth proportional
10. Extremes

a. A statement of the equality of two ratios.
b. Polygons having corresponding angles equal and corresponding sides proportional.
c. The last term of a proportion.
d. Two quantities having a common unit of measure.
e. The result of dividing one quantity by another of the same kind.
f. The second and third terms of a proportion when equal.
g. The first and last terms of a proportion.
h. By corresponding sides of similar triangles.
i. By proving a proportion.
j. The path of a moving point.

Judging the Correctness of a Converse (*10 min.*)

Read the theorem given below and think the statement of its converse. If the converse is true, write T, if false, write F.

1. A line bisecting two sides of a triangle is parallel to the third side and equal to half the third side.
2. Two triangles similar to a third triangle are similar.
3. Right triangles are similar if an acute angle of one equals an acute angle of the other.
4. The diagonals of a trapezoid divide each other into proportional segments.
5. Parallel lines cut off proportional segments on two transversals.

Investigation Problem. In the figure $\angle ACB$ is a right angle, and CD is perpendicular to AB. How many triangles are there in the figure? How many of them appear to be similar? If you wished to prove AC a mean proportional between two other lines, what triangles must you prove similar? Of what two triangles is AC a side? Are they similar? If so, complete the proportion $\frac{AD}{AC} = \frac{AC}{?}$.

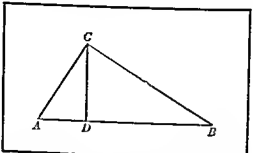
Can you prove another proportion in which BC is the mean proportional? One in which DC is the mean proportional?

Investigation Problem. In your work in arithmetic and algebra, you have used the well-known relation of the sides of the right triangle, "The square of the hypotenuse equals the sum of the squares of the other two sides." Let us try to discover a proof for this theorem. Using the figure of the last problem, and the results which you obtained there, what does $\overline{AC}^2$ equal? What does $\overline{BC}^2$ equal? Now what would you do to obtain the value of $\overline{AC}^2 + \overline{BC}^2$? Complete the following equations: $\overline{AC}^2 = AD. \ldots$ $\overline{BC}^2 = DB. \ldots$ Why? $\overline{AC}^2 + \overline{BC}^2 = (AD + DB). \ldots$ $AD + DB = \ldots$ Why? $\overline{AC}^2 + \overline{BC}^2 = \ldots$ Why?

State the converse of this theorem. Do you think it is true?

Proposition 8

225. *In a right triangle, the altitude on the hypotenuse is the mean proportional between the segments of the hypotenuse, and each leg is a mean proportional between the hypotenuse and the segment adjacent to that leg.*



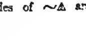
Given: [$\triangle ABC$]; $\angle ACB$ a right angle and $CD \perp AB$.

To prove: $\dfrac{AD}{CD}=\dfrac{CD}{DB}$, $\dfrac{AD}{AC}=\dfrac{AC}{AB}$, and $\dfrac{DB}{CB}=\dfrac{CB}{AB}$.

Proof:

Statements	Reasons
1. In $\triangle ABC$ and ACD, $\angle A = \angle A$.	1. Iden.
2. $\triangle ACD$ is a rt. $\triangle$.	2. $CD \perp AB$.
3. $\triangle ACD \sim \triangle ABC$. In like manner $\triangle BCD \sim \triangle ABC$.	3. § 217.
4. $\triangle ACD \sim \triangle BCD \sim \triangle ABC$.	4. § 218.
5. $\dfrac{AD}{CD}=\dfrac{CD}{DB}$, $\dfrac{AD}{AC}=\dfrac{AC}{AB}$, and $\dfrac{DB}{CB}=\dfrac{CB}{AB}$.	5. Corr. sides of $\sim\triangle$ are proportional.

226. Corollary. *A perpendicular to a diameter from any point on the circle is a mean proportional between the segments of the diameter.*



Exercises

In the right $\triangle ABC$, CD is the altitude on the hypotenuse.

1. Find CD if:

(a) $AD=2$ and $DB=8$; (b) $AD=8$ and $AB=26$;
(c) $AD=5$ and $DB=7$; (d) $AD=6$ and $DB=6$.

2. Find AD if:

(a) $CD=6$ and $DB=9$; (b) $CD=7$ and $DB=11$;
(c) $AB=10$ and $AC=6$; (d) $CD=a$ and $DB=b$.

3. Find DB if:

(a) $CD=\sqrt{60}$ and $AD=10$; (b) $BC=8$ and $AB=12$.

4. Find BC if:

(a) $AD=3$ and $DB=9$; (b) $AD=6$ and $AB=14$;
(c) $AC=9$ and $AD=3$; (d) $AD=a$ and $DB=b$.

5. Find AC if:

(a) $AD=4$ and $AB=9$; (b) $AD=c$ and $AB=d$.

6. Find AB if:

(a) $\angle A=60°$ and $AD=5$; (b) $\angle A=60°$ and $AD=b$.

7. Find DB if:

(a) $\angle A=60°$ and $AD=6$; (b) $\angle A=30°$ and $AB=10$.

8. Prove that:

(a) $\dfrac{AC}{AD}=\dfrac{CB}{CD}$; (b) $\dfrac{\overline{AC}^2}{\overline{BC}^2}=\dfrac{AD}{DB}$;
(c) $CD\times AB=AC\times BC$.

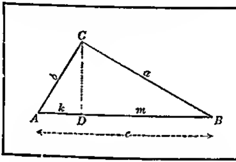
9. In $\triangle ABC$, $CD\perp AB$ and $\dfrac{AD}{AC}=\dfrac{AC}{AB}$. Then $\angle ACB$ is a rt. $\angle$.

10. In $\triangle ABC$, AB remains fixed in length and position, but rt. $\angle C$ moves toward A. What change takes place in the length of AD? AC? CD? DB? CB?

11. In right $\triangle ABC$, CD is the altitude on the hypotenuse. If CD grows longer while AD remains unchanged, what change takes place in DB? If CD remains unchanged, but AD grows shorter, what change takes place in DB?

Proposition 9

* **227.** *In a right triangle, the square of the hypotenuse equals the sum of the squares of the legs.*



Given: $\triangle ABC$ with right angle at C.
To prove: $a^2+b^2=c^2$.

Proof:

Statements	Reasons
1. Draw $CD \perp AB$. Let $AD=k$, and $DB=m$.	1. § 64.
2. $\frac{m}{a}=\frac{a}{c}$, and $\frac{k}{b}=\frac{b}{c}$.	2. § 225.
3. $a^2=mc$ and $b^2=kc$.	3. § 200.
4. $a^2+b^2=(m+k)c$.	4. Ax. 3.
5. $m+k=c$.	5. Ax. 7.
6. $a^2+b^2=c^2$.	6. Ax. 1.

228. Corollary. *The square of a leg of a right triangle equals the square of the hypotenuse minus the square of the other leg.*

Class Exercises

1. Find the hypotenuse of a right triangle whose legs are: (*a*) 6 and 8; (*b*) 5 and 12; (*c*) 10 and 10; (*d*) p and q.

2. Find the other leg of a right triangle, if the hypotenuse and one leg are respectively: (*a*) 20 and 12; (*b*) 10 and 7; (*c*) r and s.

3. Find the diagonal of a square whose side is 10 ft.

4. The base and altitude of a rectangle are respectively 6 and 8. Find its diagonal.

5. If the base of a rectangle is 12 and its diagonal is 13, find its altitude.

6. Find the altitude of an equilateral triangle whose side is 16.

7. Find the altitude of an equilateral triangle whose side is s.

8. Are the following triangles right triangles, and why, if the sides are: (a) 4, 12, and 13? (b) 3, 4, and 5? (c) 17, 12, and 25? (d) 7, 8, and 9?

9. Find the altitude of an isosceles triangle whose base is 10 and whose legs are each 13.

10. Find the legs of an isosceles right triangle whose hypotenuse is 10.

11. A tangent from a point to a circle is 24, and the radius is 10. Find the distance of the point from the center.

12. A point is 10 in. from the center of a circle whose radius is 6 in. Find the length of the tangent from the point to the circle.

13. In a circle whose radius is 20 in., what is the length of a chord 12 in. from the center?

14. If a chord 16 in. long is 15 in. from the center, what is the radius of the circle?

15. The distance from the center of a circle to a chord 10 in. long is 12 in. Find the distance from the center to a chord 24 in. long.

16. The bottom of a ladder 17 ft. long is 8 ft. from a wall. How high is the top of the ladder above the ground?

17. A ladder 25 ft. long reaches a window 20 ft. high on one side of a street. When turned over upon its foot, it reaches another window 24 ft. high on the opposite side of the street. Find the distance between the two windows.

18. A tree 64 ft. high is cut through until it falls over without breaking entirely free from the stump. If the topmost part of the

tree strikes the ground 48 ft. from the base of the stump, find the height at which it was cut.

19. The perimeter of a rectangle is 14 ft. and its diagonal is 5 ft. Find the sides of the rectangle.

20. The legs of a right triangle are 5 and 12 respectively. Find the altitude on the hypotenuse and the segments of the hypotenuse made by the altitude.

21. If the legs of a right triangle are 12 and 9, find the altitude on the hypotenuse.

22. If the diagonals of a rhombus are 10 and 24, find its side.

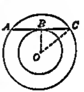
23. If a side of a rhombus is 10 and the longer diagonal is 16, find the other diagonal.

24. If two adjacent sides of a parallelogram are 5 and 12, and a diagonal is 13, is the parallelogram a rectangle? Why?

25. In a right triangle whose hypotenuse is 30 and one of whose legs is 18, find the altitude on the hypotenuse.

26. Find the side of a square whose diagonal is 10.

27. If the diameters of two circles, which have the same center, are 20 and 16, find the length of the chord of the larger circle which is tangent to the smaller.

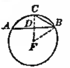


28. Find the side of an equilateral triangle whose altitude is 8.

Optional Exercises

29. The chord of an arc is 42, and the radius is 29. Find the chord of half the arc.

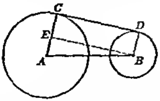
30. A flag pole 48 ft. high and a post 8 ft. high are 42 ft. apart. How long a rope will be needed to reach from the top of the pole to the top of the post?



31. Point A is 4 ft. from a circle, and the length of the tangent from A to the circle is 10 ft. Find the diameter of the circle.

32. Find the length of a tangent to a circle, whose radius is 8 in., from a point 9 in. from the circle.

33. Two circles with radii 11 ft. and 2 ft. respectively, have their centers 15 ft. apart. Find the length of the common external tangent.



34. A side of an isosceles right triangle is 4 in. Find the length of the altitude on the hypotenuse.

35. Find the radius of a circle inscribed in an equilateral triangle whose side is 16.

36. Two parallel chords of a circle, on the same side of the center, are 6 in. and 8 in. long, and 1 in. apart. Find the radius of the circle.

37. If an altitude is drawn to the hypotenuse of a right triangle, the segments of the hypotenuse are proportional to the squares of the legs.

38. How far away can the water be seen from an airplane one mile above the surface of the ocean, if the radius of the earth is 4,000 miles?

39. Two equal circles, whose radii are 10 in., have their centers 16 in. apart. Find the length of their common chord.

40. The hypotenuse of a right triangle is 12 and one angle is 30°. Find the legs.

41. A leg of a right triangle is 10. Find the other two sides, if the angle opposite 10 is (a) 30°; (b) 60°; (c) 45°.

42. The hypotenuse of a right triangle is 16 and one angle is 45°. Find the legs.

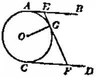
43. When the sun's rays strike the ground at an angle of 60° with the horizontal, how high is a building which casts a shadow 50 ft. long?

44. The bases of a trapezoid are 10 and 22, and the longer base forms 60° angles with both legs. Find the altitude.

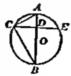
Honor Work

45. If, in $\triangle ABC$, $AD \perp BC$, prove that $\overline{AC}^2 - \overline{AB}^2 = \overline{DC}^2 - \overline{BD}^2$.

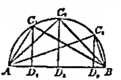
46. If AB and CD are parallel tangents, and EF is tangent at G, prove that radius OG is a mean proportional between EG and OF.



Ex. 46.



Ex. 47.



Exs. 48, 49, 50, 51.

47. In $\odot O$, diameter AB is perpendicular to chord CE. If $OB = 13$, and $OD = 5$, find CE, AC, and CB.

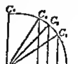
In the right $\triangle ABC$ with CD the altitude on the hypotenuse:

48. If point D moves along AB from A to B, what is the locus of point C? Prove your answer.

49. If AB is 12, find $\angle A$ when AD equals (a) 3; (b) 6; (c) 9. How does the size of $\angle A$ change as D moves from A to B?

50. If AB is 12, find AC when AD is (a) 3; (b) 6; (c) 9. When AD doubles and triples its length does AC double and triple?

51. If $\angle A$ remains 60°, find the length of AC when AD equals (a) 3; (b) 6; (c) 9. When $\angle A$ remains constant, does AC double and triple according as AD doubles and triples?



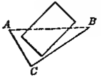
Ex. 52.

52. If $AC = 12$, find CD when $\angle A$ equals (a) 30°; (b) 45°; (c) 60°. What can you say of the length of CD as $\angle A$ approaches 90°? Also find AD for each angle given.

53. In $\triangle ABC$, $\angle C$ is a right angle. Construct $\angle A$, if $\frac{BC}{AB} = \frac{1}{3}$; if $\frac{BC}{AC} = \frac{2}{5}$.

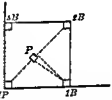
Applied Problems

54. To find the distance AB, I measure AC 112 ft. and CB 136 ft. at right angles with each other. What is the length of AB?

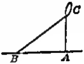


55. A baseball diamond is a square 90 ft. on a side. How far is it from home plate to second base? If the pitcher's box is 60.5 ft. from home plate, how far is it from second base?

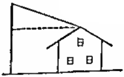
56. What is the distance from the pitcher's box to first base?



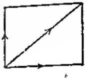
57. A surveyor constructs a perpendicular AC to AB at A, in the field, as follows. He first measures AB 40 ft. Then one man holds the zero end of the tape at A, another holds the 100 ft. mark at B, and a third holding the 30- and 50-ft. marks together, walks out until the tape is pulled tight at C. Prove that AC is the required perpendicular. (He leaves the loop at C because a steel tape would break if bent at the acute $\angle ACB$.)



58. Frank Carlton wants to put up an antenna for his radio set. He decides to run it from a pole 55 ft. in height which is 72 ft. from his house to the gable of the house which is 25 ft. from the ground. How many feet of copper wire must Frank buy for his antenna from the top of the pole to the vertex of the gable? The house is 20 ft. wide.



59. A boat travels 40 ft. across a river while the current carries it 30 ft. down stream. Draw the path of the boat, and find the distance it has moved.



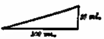
60. A man walks at the rate of 5 mi. an hour across the deck of a boat which is traveling 12 mi. an hour. What is his actual speed?

61. A force of 60 lb. is pulling directly north and a force of 80 lb. directly east. If the resultant force is represented by the diagonal of the parallelogram whose sides are proportional to the given forces, find its amount.

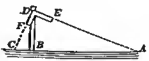
62. A railway passenger notices that while the train is standing at the station, the raindrops run vertically down the windows, but when the train is traveling 40 ft. per second, the rain makes an angle of 30° with the vertical. How many feet per second are the raindrops falling?

63. As the speed of the train increases, in Ex. 62, the angle that the raindrops make with the vertical gradually increases until it becomes 45°. What is the speed of the train now?

64. Charles Pearce is on a trip when he receives news that makes him want to reach home in the shortest time possible. He can travel by train 100 mi. at 30 mi. an hour, and then 30 mi. at right angles to the railroad by trolley at 12 mi. an hour; or he can go by automobile in a straight line to his home at 20 mi. an hour. Which route should Mr. Pearce choose?

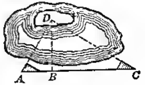


65. To find the distance AB, I fasten a carpenter's square to a post 5 ft. high at D, and aim along the edge DE at point A. Then without moving the square, I aim along DF and note the point C. If BC is $1\frac{1}{4}$ ft., what is the distance AB?

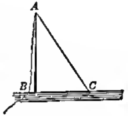


66. To find the distance from a classroom window B to the house directly across the street at C, the class measured DA perpendicular to DC from another window D, and then measured from B to the point A where CB produced met DA. If AB is found to be 2 ft. and AD 10 ft., what is the distance BC?

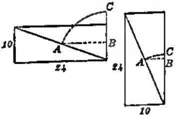
67. To find the distance from B on the shore of a pond to an island D, Jack Williams walks along BA perpendicular to BD until he reaches a point A, where, looking along the hypotenuse of a cardboard right triangle, he can see B, and looking along a leg of the triangle, he can see D. Jack next walks along BC and repeats the operation with the other acute angle of his triangle. He then finds that AB is 10 ft. and BC 40 ft. What is the distance BD?



68. A rope, fastened to the top of a flag pole and hanging close to the pole, is of such length that 10 ft. of it lies on the ground. Dorothy takes the end of the rope and pulls it out to point C, 30 ft. from the foot of the pole, where it is just long enough to reach the ground. How high is the pole?



69. The work necessary to overturn an object is the product of its weight by the height BC which its center of gravity A must be raised. A block of stone is 10 in. by 24 in. and weighs 600 lbs. What work in foot-pounds is done in overturning it, if it stands with the 24-in. edges vertical? If with the 24-in. edges horizontal?



Space Geometry (*Optional*)

In the rectangular box shown on the next page:

1. How many right angles can you find at the vertex E?

2. How many degrees are there in $\angle AEG$? In $\angle EFG$?

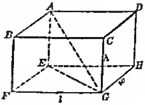
3. (*a*) If you knew the lengths of EF and FG, how could you find the length of EG?

(*b*) Find EG when $EF=6$ and $FG=8$.

4. If you knew EG and AE, how could you find AG?

(b) Find AG if $AE=5$ and $EG=12$.

5. Find AG if $AE=3$, $EF=4$ and $FG=5$.



6. Find the diagonal of the rectangular box whose dimensions are:

(a) Length 4, width 3 and height 12.

(b) Length 8, width 6 and height 5.

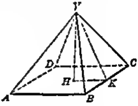
(c) Length 10, width 5 and height 4.

7. If the length, width, height and diagonal of a rectangular box are l, w, h and d, prove that $d^2=l^2+w^2+h^2$.

8. Find the diagonal of a cube whose edge is 10.

9. Show that the diagonal of a cube, whose edge is e, is $e\sqrt{3}$.

The base of this pyramid is a square and the altitude VH cuts the middle point of the base.



10. If each side of the base is 6 and the altitude is 4, find the slant height VK.

Note: Both VK and HK are $\perp BC$.

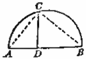
11. Prove that the edges VA and VC are equal.

12. If each side of the base is 6 and the altitude is 4:

(a) Find the length of VC by first finding HC.

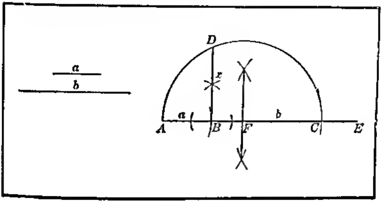
(b) Find the length of VC by first finding VK and KC.

Investigation Problem. What propositions have you learned in which a segment is proved a mean proportional to two other segments? CD is a mean proportional to what segments? Why? If two segments were given, could you construct a figure like that shown here so as to find the mean proportional between them? Make this construction showing all arcs before looking at Proposition 10.



Proposition 10

229. *A mean proportional between two given line segments can be constructed.*



Given: Line segments a and b.

To prove: x can be constructed so that $\frac{a}{x}=\frac{x}{b}$.

Construction: STATEMENTS	REASONS
1. On AE, take $AB=a$ and $BC=b$.	1. Post. 3.
2. Bisect AC at F.	2. § 63.
3. With F as center and FA as radius, construct a semicircle on AC.	3. Post. 3.
4. At B, construct $BD\perp AC$, meeting the semicircle at D.	4. § 64.
Then $BD=x$.	
Proof:	
1. $\frac{a}{x}=\frac{x}{b}$.	1. § 226.

1. Construct the mean proportional between two line segments 1 in. and 4 in. long. Check the construction algebraically.

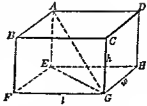
2. Given two line segments a and b, construct $x=\sqrt{ab}$. $x=\sqrt{6ab}$.

3. Given a line segment a construct $x=a\sqrt{3}$

4. If you knew EG and AE, how could you find AG?

(b) Find AG if $AE=5$ and $EG=12$.

5. Find AG if $AE=3$, $EF=4$ and $FG=5$.



6. Find the diagonal of the rectangular box whose dimensions are:

(a) Length 4, width 3 and height 12.

(b) Length 8, width 6 and height 5.

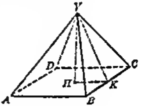
(c) Length 10, width 5 and height 4.

7. If the length, width, height and diagonal of a rectangular box are l, w, h and d, prove that $d^2=l^2+w^2+h^2$.

8. Find the diagonal of a cube whose edge is 10.

9. Show that the diagonal of a cube, whose edge is e, is $e\sqrt{3}$.

The base of this pyramid is a square and the altitude VH cuts the middle point of the base.



10. If each side of the base is 6 and the altitude is 4, find the slant height VK.

Note: Both VK and HK are $\perp BC$.

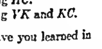
11. Prove that the edges VA and VC are equal.

12. If each side of the base is 6 and the altitude is 4:

(a) Find the length of VC by first finding HC.

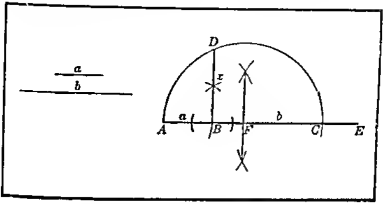
(b) Find the length of VC by first finding VK and KC.

Investigation Problem. What propositions have you learned in which a segment is proved a mean proportional to two other segments? CD is a mean proportional to what segments? Why? If two segments were given, could you construct a figure like that shown here so as to find the mean proportional between them? Make this construction showing all arcs before looking at Proposition 10.



Proposition 10

229. *A mean proportional between two given line segments can be constructed.*



Given: Line segments a and b.

To prove: x can be constructed so that $\frac{a}{x}=\frac{x}{b}$.

Construction: STATEMENTS	REASONS
1. On AE, take $AB=a$ and $BC=b$.	1. Post. 3.
2. Bisect AC at F.	2. § 63.
3. With F as center and FA as radius, construct a semicircle on AC.	3. Post. 3.
4. At B, construct $BD \perp AC$, meeting the semicircle at D.	4. § 64.
Then $BD=x$.	
Proof:	
1. $\frac{a}{x}=\frac{x}{b}$.	1. § 226.

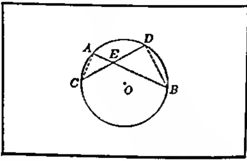
1. Construct the mean proportional between two line segments 1 in. and 4 in. long. Check the construction algebraically.

2. Given two line segments a and b, construct $x=\sqrt{ab}$. $x=\sqrt{6ab}$.

3. Given a line segment a construct $x=a\sqrt{3}$

Proposition 11

* 230. *If two chords intersect, the product of the segments of one equals the product of the segments of the other.*



Given: $\odot O$, chords AB and DC intersecting at E.
To prove: $AE \times EB = CE \times ED$.

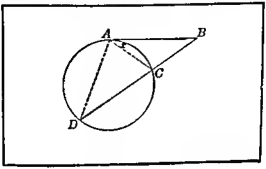
Proof:

Statements	Reasons
1. Draw AC and BD.	1. Post. 1.
2. In $\triangle ACE$ and DBE, $\angle C = \angle B$ and $\angle A = \angle D$.	2. § 167.
3. $\triangle ACE \sim \triangle DBE$.	3. § 216.
4. $\frac{AE}{ED} = \frac{CE}{EB}$.	4. § 223.
5. $AE \times EB = CE \times ED$.	5. § 200.

1. Construct a line whose ratio to a given line shall be $\sqrt{2}$ to 1.
2. Construct two lines in the ratio $\sqrt{2}$ to $\sqrt{3}$.
3. Divide a given line into segments in the ratio 1 to $\sqrt{5}$.
4. Given two lines a and b, construct $x = \sqrt{a^2 + b^2}$.
5. Given two lines a and b, with $a > b$, construct $x = \sqrt{a^2 - b^2}$.
6. Construct a right triangle, given a leg and the altitude on the hypotenuse.
7. Given the perimeter, construct a triangle similar to a given triangle.
8. Construct a triangle, given two angles and the perimeter.

PROPOSITION 12

231. *If a tangent and a secant are drawn from a point outside a circle, the tangent is the mean proportional between the secant and its external segment.*



Given: $\odot ADC$, tangent AB and secant BD.

To prove: $\dfrac{BD}{BA}=\dfrac{BA}{BC}$.

Proof:

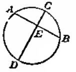
STATEMENTS	REASONS
1. Draw AC and AD.	1. Post. 1.
2. In $\triangle ABC$ and ABD, $\angle B = \angle B$.	2. Iden.
3. $\angle D$ is measured by $\frac{1}{2}\widehat{AC}$.	3. § 165.
4. $\angle x$ is measured by $\frac{1}{2}\widehat{AC}$.	4. § 170.
5. $\angle D = \angle x$.	5. Ax. 2.
6. $\triangle ABC \sim \triangle ABD$.	6. § 216.
7. $\dfrac{BD}{BA}=\dfrac{BA}{BC}$.	7. § 223.

232. Corollary. *If two secants are drawn from a point outside a circle, the product of one secant and its external segment equals the product of the other secant and its external segment.*

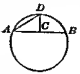
Can you construct a mean proportional to two line segments by a method based on Proposition 12.

Class Exercises

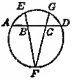
1. If $AE=10$, $EB=8$, and $CE=6$, find ED.
2. If $AE=EB$, $CE=8$, and $ED=18$, find AB.
3. If $AE=CE$, prove that $DE=BE$.



Exs. 1, 2, 3, 4



Ex. 10.



Ex. 11.

4. If CD is a diameter perpendicular to AB, $AB=12$, and $CE=4$, find CD.

5. The chord of an arc is 24 in. and the middle point of the arc is 9 in. from the chord. Find the diameter of the circle.

6. A chord is drawn through a point 5 in. from the center of a circle, whose radius is 13 in. Find the product of the segments of the chord.

7. The distance from the center of a chord 12 ft. long to the center of its arc is $2\frac{1}{2}$ ft. Find the radius of the circle.

8. If two chords AB and CD intersect at E, making $AE=m$, $EB=n$, and $CE=p$, find ED.

9. Two chords intersect inside a circle. The segments of one are each 4, and the total length of the other is 10. Find the length of its segments.

10. C and D are the middle points of a chord and its arc. If $AD=9$ and $CD=3$, what is the diameter of the circle?

11. If chord AD is trisected at B and C, and $EB=GC$, prove that $\triangle FBC$ is isosceles.

Optional Exercises

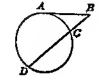
12. One of two secants meeting outside a circle is 12.5 in., and its external segment is 4 in. The other secant is bisected by the circle. Find the length of the second secant.

13. If tangent AB equals 8 in. and BC equals 4 in., find DC.

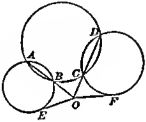
14. If $BC = 3$ in. and $CD = 9$ in., find the length of tangent AB.

15. If tangent AB equals 15 in., and CD equals 16 in., find BC and BD.

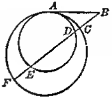
16. If tangent AB equals 12 in., and the distance from B to the center of the circle is 13 in., find the radius.



Exs. 13, 14, 15.



Ex. 22.



Ex. 24.

17. A tangent and a secant are drawn to a circle from an external point. The tangent is 14 in. long, and the whole secant is four times its external segment. Find the length of the secant.

18. From a point 21 in. from the center of a circle, whose radius is 15 in., a secant is drawn. Find the product of the whole secant and its external segment.

19. A point A is 4 ft. from a circle, and the length of the tangent from A to the circle is 10 ft. Find the diameter of the circle.

20. If two circles intersect, their tangents from any point on the common chord produced, are equal.

21. If two circles intersect, the common chord produced bisects the common tangents.

22. A $\odot ABCD$ is cut by a second circle at A and B, and by a third circle at C and D. The lines AB and DC, produced, meet at O. Prove that the tangents OE and OF are equal.

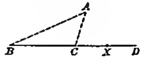
23. If you are standing on a cliff 500 ft. high overlooking the ocean, how far out on the water can you see, if the earth's diameter be taken as 8,000 mi.?

24. If two circles are tangent at A, and AB is their common tangent, prove that $BC \times BF = BD \times BE$

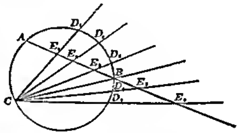
Honor Work

25. If B and C are two points on the line BD, and A is a point not on the line, construct a point X on CD, so that $BX \times CX = \overline{AX}^2$.

(Suggestion: Circumscribe a circle about $\triangle ABC$.)



26. If the chord CD turns around point C until E coincides with B, (a) describe the motion of D. (b) What is the value of the product $CE \times ED$ when E arrives at B? (c) What is then the value of $AE \times EB$? (d) Does $AE \times EB$ still equal $CE \times ED$?



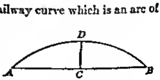
27. If CD turns still farther until AB and CD intersect outside the circle, is it still true that $AE \times EB = CE \times ED$? Give a reason for your answer.

Applied Problems

28. A circular arch of masonry having a radius of 25 ft. rests on two stone piers, that are 40 ft. apart. Find the height of the center of the arch above the level of the top of the piers.

29. The piers of a bridge in the form of a circular arch are 200 ft. apart, and the highest point of the arch is 25 ft. higher than the piers. Find the radius of the arch.

30. A and B are two points on a railway curve which is an arc of a circle. If the chord AB is 200 ft., and the shortest distance from the middle point of the curve to the chord is 15 ft., find the radius of the curve.

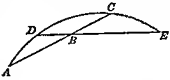


31. In constructing a railway curve having a radius of 400 ft. between A and B, which are two points 300 ft. apart, a surveyor

first locates the middle point D. Find the number of feet from the middle point C of chord AB which he must measure.

32. A bridge, 1080 ft. long, is built on 12 equal arches, each arch rising 25 ft. above its supporting pier. Find the radius of the arches.

33. An engineer, wishing to locate other points such as E on a railway curve, of which the part AC is already laid out, measures AB, DB and BC. If $AB = 200$ ft., $DB = 80$ ft., and $BC = 120$ ft., how far from B is the point E?



Photograph from Ewing Galloway.

THE USE OF CIRCULAR ARCHES IN BRIDGE CONSTRUCTION.
Notice both the large arches and the smaller ones. This is a picture of the Tunkhannock Viaduct of the Lackawanna Railroad.

A Self-Measuring Test

1. State two proportions which are true in the right triangle when the altitude is drawn on the hypotenuse.

2. How can you find the hypotenuse of a right triangle when the legs are known?

3. How can you find a leg of a right triangle when the hypotenuse and the other leg are known?

4. State two propositions in which a mean proportional is mentioned.

5. State a proposition on which the construction of the mean proportional depends.

6. On what proposition in Book Two does the construction of the mean proportional depend?

7. Give a proposition which states that the product of two lines equals the product of two other lines.

8. Give two methods of proving triangles similar.

9. How many similar triangles are formed when an altitude is drawn to the hypotenuse of a right triangle?

10. State two propositions in Book Three in which the measurement of the inscribed angle is used.

Completion Test (*10 min.*)

1. If the hypotenuse of an isosceles right triangle is 10, then one of the equal legs is ....

2. A diameter of a circle perpendicular to a chord is divided by the chord into segments of 2 in. and 18 in. The length of the chord is ....

3. The bases of an isosceles trapezoid are 20 and 30 and the angles at the ends of the longer base are each 45°. The altitude is ....

4. If a tangent and a secant, drawn from the same point to a circle, are 6 and 18 respectively, the external segment of the secant is ....

5. If two parallel chords of a circle are each 24 and the distance between them is [illegible], the radius of the circle is ....

Numerical Test (*10 min.*)

1. If the diagonals of a rhombus are 6 and 8, find a side.
2. Find the altitude of an equilateral triangle whose side is 8.
3. The altitude on the hypotenuse of a right triangle divides the hypotenuse into segments 5 and 4. Find the shorter leg.
4. In a circle whose radius is 10, find the distance from the center to a chord whose length is 16.
5. In a right triangle, one leg is 8 and the hypotenuse is 10 Find the shorter segment of the hypotenuse made by the altitude on it.

True-False Test (*10 min.*)

In the right triangle ABC, CD is the altitude on the hypotenuse. Tell which of the following proportions are true:

1. $\frac{AB}{AC}=\frac{AC}{AD}$.	4. $\frac{AC}{CD}=\frac{CD}{DB}$.	7. $\frac{DB}{CB}=\frac{CD}{AC}$.
2. $\frac{AC}{CB}=\frac{AD}{DB}$.	5. $\frac{DB}{CB}=\frac{CB}{AC}$.	8. $\frac{AD}{CD}=\frac{CD}{DB}$.
3. $\frac{AD}{CD}=\frac{AC}{CB}$.	6. $\frac{DB}{CB}=\frac{CB}{AB}$.	9. $\frac{AC}{CD}=\frac{CD}{CB}$.

Multiple-Choice Test (*10 min.*)

From the answers given, select that one which makes the statement true.

1. The altitude on the hypotenuse of a right triangle cuts the figure into two triangles which are, (*a*) isosceles; (*b*) similar; (*c*) equilateral; (*d*) congruent.
2. A chord ABC turns around a given point B inside a circle. If the segment AB is increasing, the segment BC is (*a*) increasing; (*b*) equal to AB; (*c*) decreasing; (*d*) constant in length.
3. A diameter cuts another chord into two segments 2 in. and 6 in. One segment of the diameter is 1 in. The length of the diameter is, (*a*) 12; (*b*) 7; (*c*) 8; (*d*) 13.

4. Two chords AB and CD intersect at E inside a circle. AB is 6, EB is 4 and CE is 3. If a length $EX=7$ is laid off on ED, extended if necessary, the point X is, (*a*) inside; (*b*) on; (*c*) outside; (*d*) at the center of, the circle.

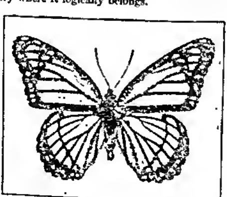
5. A tangent and a secant are drawn to a circle from the same point. If the tangent is 6 and the external segment of the secant is 2, the secant is, (*a*) 18; (*b*) 20; (*c*) 12; (*d*) 14.

233. Approximate measurement.[1] It is possible to think of a line as exactly equal to another line, or exactly 8 in.

[1] The material included under the headings, Approximate Measurements and Functions of Angles, pages 266-303, may be omitted from the course in plane geometry without interfering with the logical continuity of the subject by teachers wishing to confine themselves to the traditional course.

The Report of the National Committee on Mathematical Requirements places it, together with other important topics, in the required work of the ninth school year, stating that "it is the opinion of the committee that the material included in this chapter should be required of all pupils. It includes mathematical knowledge and training which is likely to be needed by every citizen." It is also given the same position and emphasis by the College Entrance Examination Board.

Since however few schools at present give this work during the ninth school year, it is strongly advised that it be included in the course in plane geometry where it logically belongs.



THE BUTTERFLY IS A BEAUTIFUL EXAMPLE OF SYMMETRY IN NATURE

Most animals and some plants are symmetrical.

long, and to think of triangles as exactly congruent, just as it is possible to think of a point as having no size or of a line as having no width. Therefore a geometry based on such concepts is an exact science. The conclusions drawn are absolutely exact if the assumptions made are exact. For example, if we knew that two sides of a triangle were exactly equal, then it would follow that the angles opposite those sides would be exactly equal.

But because we are unable to make instruments that are absolutely accurate, and to read them beyond a certain degree of exactness, lines are never known to be exactly equal. All physical measurements are only approximately true. With an ordinary ruler, we cannot measure the length of this page more accurately than to the hundredth of an inch. We read the length as 7.25 inches, if, in our judgment, it is more than 7.245 and less than 7.255, that is, if it is nearer to 7.25 than to 7.24 or 7.26. But we cannot read accurately enough to tell if it is 7.249 or 7.252. Therefore we say that the length is 7.25 inches to the nearest hundredth of an inch.

If we used a more accurate instrument than the ruler, we might determine that the length was 7.252 and not 7.251 or 7.253, but we would still be unable to tell if it differed from 7.252 by 1 or 2 ten-thousandths of an inch. All that our more accurate instrument can do is to move the uncertainty to a different decimal place.

Significant figures. When we measured with the more accurate instrument, if we had found that the result was nearer to 7.25 than to 7.251 or to 7.249, we would have written the result 7.250 in. This zero does not change the result, but it shows that we have measured the length more accurately than to the hundredth of an inch. Such a number, which indicates the degree of accuracy of a measurement, is called a *significant figure.* In the first measurement, the 7, 2. and 5 are all significant figures, for the result is.

accurate to the hundredth of an inch, but if zeros were written after the 5, they would not be significant.

Very often zeros are not significant figures, but are used only to determine the decimal point. When a man states, for example, that he expects to save \$1,000 next year, he does not mean that he will not save \$1,143.57, but only that he does not expect to differ from \$1,000 by many hundred dollars. In this case, the zeros merely determine the decimal point; they show that he does not expect his savings to be around \$10,000 or \$100, so the number 1,000 has only one significant figure. Similarly, the distance from the earth to the moon, 238,000 miles, has only three significant figures, for it may differ from the exact distance by three or four hundred miles. A number, which has zeros to determine the decimal point, is called a *round number*.

All figures except zeros are usually significant. We are accustomed to say that a distance is *about* 200 miles when we know it is 198 miles, but we do not say that it is about 198 miles when we know that it is 200 miles.

234. Computation with approximate numbers. If we find, by measurement, that the side of a square is 5.4 in., correct to the nearest tenth of an inch, then the diagonal will be $5.4\sqrt{2}$. Now $\sqrt{2}$ is 1.41421 correct to five decimal places, which gives us 7.636734 as the length of the diagonal. However 5.4 in. to the nearest tenth of an inch only tells us that the side of the square is between 5.35 and 5.45 in., and may differ from the exact length as much as .04 in. The length of the diagonal, then, computed from this number, may differ from the true length by more than .04 in. Consequently it would be not only useless but misleading to give the result 7.636734, when even the three in the hundred's place may be incorrect. Since 5.4 is accurate to only two significant figures, the length of the diagonal cannot be depended on to more than two significant figures. As a

rule, in working with approximate measurements, we carry the result to only as many significant figures as the least accurate of the measurements used in obtaining it.

Exercises

Assuming that these numbers are approximate measurements, find the answers to such a degree of accuracy as the data justify.

1. The dimensions of a rectangle are 18 in. and 1.402 in. Find the diagonal.

2. On a map, the distance from A to B is 2.4 in., and from B to C 4.83 in. If the distance from A to B is 267.6 mi., find the distance from B to C.

3. A chord of a circle is 38.42 ft. and the distance from its middle point to the middle point of its arc is 2 ft. Find the radius of the circle.

4. Find a mean proportional between 1.8 and 2.374.

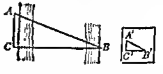
5. Using ruler and protractor, draw a triangle whose base is about 3.5 in., and whose base angles are 27° and 90°. Measure the other two sides and find their ratio.

Numerical Trigonometry

235. Functions of angles. Thus far in our course we have learned many methods by which the length of a line can be found without actually measuring it. In Book One we found the length of a line by measuring another line which was its equal in length. This was useful because it saved measuring through a swamp or across a river, but the actual length measured was the same. Then in Book Three, we computed the length of a longer line by measuring three shorter lines and by using proportion. This is obviously a better method for many purposes, for not only can we find results more accurately by multiplication and division than we can by actually measuring a long distance, but the amount of work is greatly reduced. Can the work be reduced still further? Consider the following problem:

Paul wished to find the distance across the Otter Creek

from C to B, so he measured a distance of 100 ft. in the direction CA at right angles to CB, and then he found with a protractor that $\angle A$ was 78°. He next drew carefully on paper a right $\triangle A'B'C'$, having $\angle A'$ equal to 78°, and measured $C'B'$ and $A'C'$. He found that, if he made $A'C'$ 2 in. long, $C'B'$ was about 9.4 in. long. Paul concludes that, since $C'B'$ is 4.7 times as long as $A'C'$, CB must be 4.7 times as long as AC or 470 ft., for $\triangle ABC$ and $A'B'C'$ are similar.

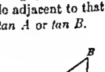


Now Paul measured only one line on the ground and two short lines on paper. And he measured the lines $C'B'$ and $A'C'$ on paper only to determine the ratio of CB to AC. Evidently, if he had known this ratio beforehand, he would have needed to measure only one line. Since all right triangles, having an $\angle A$ equal to 78°, are similar, this ratio could be computed once for all for a 78° angle. Therefore it would not be necessary to draw a triangle on paper every time we wished to find the ratio. Then, however large or small $\triangle ABC$ is, this ratio will remain unchanged so long as the size of $\angle A$ remains unchanged. But if $\angle A$ grows larger or smaller, the ratio will change with it. This ratio is, therefore, called a *function* of $\angle A$.

236. In $\triangle ABC$, in which $\angle C$ is a right angle, the ratio of the side opposite an acute angle to the side adjacent to that angle is the tangent of the angle, written *tan A* or *tan B*.

$$\text{Tan } A = \frac{a}{b} = \frac{\text{opposite leg}}{\text{adjacent leg}}$$

$$\text{Tan } B = \frac{b}{a} = \frac{\text{opposite leg}}{\text{adjacent leg}}$$



EXERCISES

1. Using a protractor, draw a right triangle having $\angle A$ equal to 10°. Measure a and b carefully and compute the value of tan 10° to the nearest hundredth.

2. From the same triangle compute tan 80°.

3. By drawing a triangle having $\angle A$ equal to 20°, compute tan 20° and tan 70°.

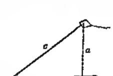
4. Using the same method, fill the blanks in the following table:

Angle	Tan	Angle	Tan
10°		50°	
20°		60°	
30°		70°	
40°		80°	

5. Find tan 45° without measuring lines.

6. Henry finds that a wire from the top of a telegraph pole touches the ground 20 ft. from the foot of the pole and makes an angle of 50° with the ground. Using your table, find the height of the pole.

237. If we know any side of a right triangle and an acute angle, we can find the other two sides by using the tangent ratio, but sometimes this method would be very inconvenient, as in the following example:



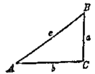
Robert is flying his kite. After letting out 100 ft. of string, he notices that the string makes an angle of 40° with the ground. Assuming that the string is straight, how high is the kite? Robert says that he could determine the height very easily if he knew the ratio of a to c.

Evidently to solve problems such as this, we need another ratio, that of the opposite side to the hypotenuse.

238. In a right triangle, the ratio of the side opposite an acute angle to the hypotenuse is the sine of the angle, written *sin A* or *sin B*.

$$\text{Sin } A = \frac{a}{c} = \frac{\text{opposite leg}}{\text{hypotenuse}}$$

$$\text{Sin } B = \frac{b}{c} = \frac{\text{opposite leg}}{\text{hypotenuse}}$$



The sine of B is called the *cosine of* A, written *cos A*.

Exercises

1. Draw a right triangle having $\angle A$ equal to 10° and measure a and c carefully. From their lengths, find sin 10° to the nearest hundredth

2. From the same triangle, find sin 80°.

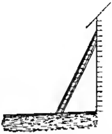
3. By drawing right triangles and measuring their sides as in the last group of exercises, fill the blanks in the following table:

Angle	Sin	Angle	Sin
10°		50°	
20°		60°	
30°		70°	
40°		80°	

4. Using the value from your table, find the height of Robert's kite.

5. Make a table for cosines like that for sines in Ex. 3. How can you find the cosine of an angle from a table giving only sines of angles?

6. John has a ladder 12 ft. long on which he wishes to climb to the eaves of his house. If, for safety, the ladder must make an angle of 70° with the ground, how high will it reach? Use your table.



239. Use of the tables. You have already made a two-place table of the sines and tangents of angles at intervals of 10°. In the same way you could have filled in these functions for angles differing by one degree. On page 295 such a table is given. In the first column is the number of degrees in the angle. In the second column, opposite the number of degrees, is the sine of the angle, and in the fourth column the tangent of the angle is given.

To find sin 37°, look down the column headed *degrees*, until you come to 37. The number .6018 found opposite 37 in the column headed *sin* is the answer.

Find tan 18°. The required answer is the number .3249, found in the *tangent* column opposite 18.

Exercises

Use the table on page 243 for the following exercises.

1. Verify the following:

   (a) $\sin 13° = .2250$
   (b) $\tan 77° = 4.3315$
   (c) $\tan 24° = .4452$
   (d) $\sin 86° = .9976$
   (e) $\cos 65° = .4226$
   (f) $\cos 20° = .9397$

2. Find the value of the following functions:

3. Verify the following angles:

(a) $\sin x = .7660 \quad x = 50°$ (c) $\tan x = 28.6363 \quad x = 88°$

(b) $\sin x = .9962 \quad x = 85°$ (d) $\tan x = .5543 \quad x = 29°$

4. Find the value of x for the following functions:

(a) $\sin x = .4540$ (d) $\sin x = \frac{1}{2}$

(b) $\tan x = .7002$ (e) $\sin x = .8090$

(c) $\tan x = 2.1445$

5. For what values of x is $\sin x$ nearly equal to $\tan x$?

6. For what value of x do $\sin x$ and $\tan x$ differ most?

7. For what value of x is $\tan x$ about twice $\sin x$?

8. Is the sine of a 60° angle twice the sine of a 30° angle? Is the tangent of 80° twice the tangent of 40°?

9. In a right triangle, the side opposite an angle of 64° is ...... times the side adjacent to the angle.

10. In a right triangle, the side opposite an angle of 22° is ...... times the hypotenuse.

11. In the right $\triangle ABC$, a is 3 and b is 4.

Tan A is ..... Tan B is ......
Sin A is ..... Sin B is ......
Side c is ...... Cos A is ......

12. Without using tables, find $\tan A$, $\tan B$, $\sin A$, and $\sin B$, if:

(a) a is 6 and b is 8

(b) a is 5 and c is 13

(c) b is 12 and c is 15

13. If $\angle A$ is 45° and a is 1, b is ...... and c is ......

14. Without using tables, find:

(a) Sin 45° and tan 45°

(b) Sin 60° and tan 60°

(c) Sin 30° and tan 30°

240. Finding distances by means of the right triangle. It is evident that if one side and the ratios of the sides of a triangle are known, the other sides can be found. Therefore, to find a side, choose that function of the known angle which is the ratio of the given side and the side to be found.

Trigonometric Table

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2°	.0349	.9994	.0349	47°	.7314	.6820	1.0724
3°	.0523	.9986	.0524	48°	.7431	.6691	1.1106
4°	.0698	.9976	.0699	49°	.7547	.6561	1.1504
5°	.0872	.9962	.0875	50°	.7660	.6428	1.1918
6°	.1045	.9945	.1051	51°	.7771	.6293	1.2349
7°	.1219	.9925	.1228	52°	.7880	.6157	1.2799
8°	.1392	.9903	.1405	53°	.7986	.6018	1.3270
9°	.1564	.9877	.1584	54°	.8090	.5878	1.3764
10°	.1736	.9848	.1763	55°	.8192	.5736	1.4281
11°	.1908	.9816	.1944	56°	.8290	.5592	1.4826
12°	.2079	.9781	.2126	57°	.8387	.5446	1.5399
13°	.2250	.9744	.2309	58°	.8480	.5299	1.6003
14°	.2419	.9703	.2493	59°	.8572	.5150	1.6643
15°	.2588	.9659	.2679	60°	.8660	.5000	1.7321
16°	.2756	.9613	.2867	61°	.8746	.4848	1.8040
17°	.2924	.9563	.3057	62°	.8829	.4695	1.8807
18°	.3090	.9511	.3249	63°	.8910	.4540	1.9626
19°	.3256	.9455	.3443	64°	.8988	.4384	2.0503
20°	.3420	.9397	.3640	65°	.9063	.4226	2.1445
21°	.3584	.9336	.3839	66°	.9135	.4067	2.2460
22°	.3746	.9272	.4040	67°	.9205	.3907	2.3559
23°	.3907	.9205	.4245	68°	.9272	.3746	2.4751
24°	.4067	.9135	.4452	69°	.9336	.3584	2.6051
25°	.4226	.9063	.4663	70°	.9397	.3420	2.7475
26°	.4384	.8988	.4877	71°	.9455	.3256	2.9042
27°	.4540	.8910	.5095	72°	.9511	.3090	3.0777
28°	.4695	.8829	.5317	73°	.9563	.2924	3.2709
29°	.4848	.8746	.5543	74°	.9613	.2756	3.4874
30°	.5000	.8660	.5774	75°	.9659	.2588	3.7321
31°	.5150	.8572	.6009	76°	.9703	.2419	4.0108
32°	.5299	.8480	.6249	77°	.9744	.2250	4.3315
33°	.5446	.8387	.6494	78°	.9781	.2079	4.7046
34°	.5592	.8290	.6745	79°	.9816	.1908	5.1446
35°	.5736	.8192	.7002	80°	.9848	.1736	5.6713
36°	.5878	.8090	.7265	81°	.9877	.1564	6.3138
37°	.6018	.7986	.7536	82°	.9903	.1392	7.1154
38°	.6157	.7880	.7813	83°	.9925	.1219	8.1443
39°	.6293	.7771	.8098	84°	.9945	.1045	9.5144
40°	.6428	.7660	.8391	85°	.9962	.0872	11.4301
41°	.6561	.7547	.8693	86°	.9976	.0698	14.3007
42°	.6691	.7431	.9004	87°	.9986	.0523	19.0811
43°	.6820	.7314	.9325	88°	.9994	.0349	28.6363
44°	.6947	.7193	.9657	89°	.9998	.0175	57.2900
45°	.7071	.7071	1.0000				

Solve this for the side to be found, and substitute the values which may be given or found from the table.

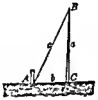
ILLUSTRATION 1. Mr. White wishes to brace an antenna pole in his back yard by fastening a wire from its top to a post 23 ft. from the foot of the pole. He finds that the angle at A is 41°. What length of wire will he need?

Here b is 23 and $\angle A$ is 41°. To find c. The ratio $\frac{b}{c}$ is the sine of $\angle B$. Therefore we must first find $\angle B$.

$$\angle B = 90° - \angle A$$
$$= 90° - 41°$$
$$= 49°$$
$$\sin B = \frac{b}{c}$$

Multiplying by c $\qquad c \sin B = b$

Dividing by sin B $\qquad c = \frac{b}{\sin B}$



Since $\qquad b$ is 23 and $\sin 49° = .76$ to two significant figures,

$$c = \frac{23}{.76} = 30$$

ILLUSTRATION 2. Chester, who is at point C, observes that an airplane is directly overhead at the same time that Tom 873 yds. away at A finds the angle of elevation ($\angle A$) to be 62°. How high is the airplane?

Here b is 873 and $\angle A$ is 62°.

We can find a by using the ratio $\frac{a}{b} = \tan A$

Then multiplying by b: $\qquad a = b \tan A$

Since we know b to three significant figures, we shall use three figures for $\tan A$. So

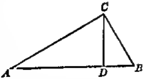
$a = 873 \times 1.88$

$= 1641.24$

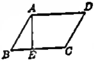
$= 1640$ yds., to three significant figures.

EXERCISES

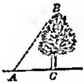
In $\triangle ABC$, $\angle C$ is a right angle.



1. Given $c=30$ and $\angle A=28°$, find b.
2. Given $c=10$ and $\angle B=53°$, find b.
3. Given $b=2.5$ and $\angle A=75°$, find a.
4. Given $a=20$ and $\angle A=35°$, find b.
5. Given $b=.42$ and $\angle B=20°$, find c.
6. Given $a=350$ and $\angle B=84°$, find c.
7. In $\triangle ABC$, $AC=34$ and $\angle A=30°$, find the altitude CD.
8. In $\triangle ABC$, altitude $CD=10$ and $\angle A=47°$, find side AC.
9. The hypotenuse AB of right $\triangle ABC$ is 18 and $\angle A$ is 18°. Find side AC and the altitude on the hypotenuse CD.



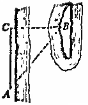
10. Find the altitude AE of $\square ABCD$, if $AB=50$ ft., $BC=80$ ft., and $\angle B=52°$.
11. At a distance of 100 ft. from its base, the angle of elevation, $\angle A$, of the top of a tree is found to be 38°. How high is the tree?



12. When the angle of elevation of the sun is 68°, find the height of a building which casts a shadow 24 ft. long.
13. When the angle of elevation of the sun becomes 33°, find the length of the shadow cast by the building in Ex. 12.
14. How far from the house must I place the foot of a ladder 14 ft. long so that it will make an angle of 65° with the ground?
15. A steep mountain road makes an angle of 36° with the horizontal How many feet have I risen when I have traveled 385 ft. up the road?

16. Two artillery officers want to locate the position of the enemy's battery by the flash of the guns. Lieut. Homer observes that $\angle BCA$ is a right angle, whereas Capt. Meyer 463 ft. from him at A finds that $\angle BAC$ is 82°. How far is the battery, B, from C?

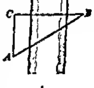
17. Girl Scouts, who planned to conduct a swimming contest from point C on the shore to point B on an island, found the distance as follows. They measured CA 340 ft. along the shore at right angles to CB, and then found that $\angle A$ was 38°. What is the distance from C to B?



18. A tower on the bank of a river is known to extend 163 ft. above the water. If the angle of elevation of the top from a point on the opposite bank is 15°, how wide is the river?

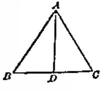
19. An astronomer at D observes that the moon is directly overhead. Six hours later, when the earth has turned so that he is at B, he finds $\angle CBA$ to be 89°. If the radius of the earth is approximately 4000 miles, how far is the moon from B?

20. John decides to run a radio antenna from his window to a tree 114 ft. away. He observes that, from his window, the angle of elevation of the point on the tree to which he intends to attach the wire is 28°. What length of wire will he need?

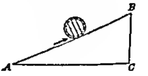


21. To find the distance CB across a river, a surveyor measured a line AC at right angles to CB, and found $\angle A$. If $\angle A = 82°$, and $AC = 120$ ft., how far is it from C to B?

22. In an isosceles $\triangle ABC$, $AB = 12$ and $\angle B = 70°$. Find the base BC and the altitude AD.



23. The force necessary to keep a barrel from rolling down an inclined plank AB, is equal to the weight of the barrel multiplied by the sine of the angle of elevation ($\angle A$). If $\angle A = 25°$, and the barrel weighs 250 lbs., what force is needed?

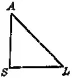


24. Find the altitude of an equilateral triangle whose side is 328.

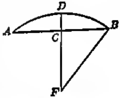
25. Find the side of an equilateral triangle whose altitude is 62.

26. When a ball rolls down an inclined plane BA, its speed at the end of t seconds is $32t \cdot \sin A$. If $\angle A = 15°$, what is its speed at the end of three seconds?

27. The angle of elevation of the top of a mountain known to be 6000 ft. high is 19°. How far is the top of the mountain from the observer?



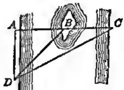
28. The captain of a ship, S, observed that a lighthouse, L, was directly east. After sailing 5 mi. north, he noticed that the lighthouse was 33° east of south ($\angle A = 33°$). How far was he from the lighthouse at first?



29. On a railway curve, an arc of 20° has a chord 340 ft. long. Find the radius of the curve.

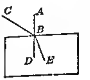
30. In a circle whose radius is 20, a chord has an arc of 100°. How long is the chord?

31. Find the number of degrees in an arc whose chord is 83.4, if the radius of the circle is 65.4



32. If the angle of elevation of the top of a tower 800 ft. away is 9°, what is the height of the tower?

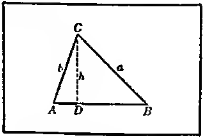
33. To find the distance from each shore to an island B, an engineer measures AD, a distance of 100 ft., perpendicular to AC. He finds that $\angle ADB = 57°$ and $\angle ADC = 74°$. Determine the distances AB and BC.



34. The index of refraction of a piece of glass is $\frac{\sin \angle ABC}{\sin \angle DBE}$, where AD is perpendicular to the surface of the glass, CB is the direction of the ray of light before entering the glass, and BE is its direction after entering. Find the index of refraction if $\angle ABC = 66°$ and $\angle DBE = 48°$.

Proposition 13

241. *The sides of any triangle are proportional to the sines of the opposite angles.*



Given: $\triangle ABC$.

To prove: $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$.

Proof:

Statements	Reasons
1. Construct $CD \perp AB$.	1. § 64.
2. $\frac{h}{b}=\sin A$, and $\frac{h}{a}=\sin B$.	2. § 238.
3. $h=b \sin A$ and $h=a \sin B$.	3. Ax. 5
4 $b \sin A=a \sin B$.	4. Ax. 2.
5. $\frac{\sin A}{a}=\frac{\sin B}{b}$.	5. § 203
Similarly, $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$.	

242. Method of attack. *Given two angles and a side of a triangle, to find the other parts.* Of the three ratios, $\frac{\sin A}{a}$, $\frac{\sin B}{b}$, $\frac{\sin C}{c}$, choose two: (1) the ratio that contains the known side, and (2) the ratio that contains the side to be determined. Solve the equation for the unknown side, and substitute the known values.

ILLUSTRATION. Given $A=62°$, $B=84°$, and $a=142$ Find b, c, and C.

$$\begin{aligned} C &= 180° - (A+B) \\ &= 180° - 146° \\ &= 34° \end{aligned}$$

To find b:

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$b \sin A = a \sin B$$

$$\begin{aligned} b &= \frac{a \sin B}{\sin A} \\ &= \frac{142 \times .995}{.883} \\ &= 160 \end{aligned}$$

To find c:

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$$c \sin A = a \sin C$$

$$\begin{aligned} c &= \frac{a \sin C}{\sin A} \\ &= \frac{142 \times .559}{.883} \\ &= 89.9 \end{aligned}$$

EXERCISES

In the following triangles, find the parts not given:

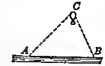
1. $A=80°$, $B=43°$, $c=12.6$
2. $B=35°$, $C=63°$, $b=312$
3. $A=45°$, $B=74°$, $b=66$
4. $A=31°$, $B=86°$, $a=13.7$
5. $B=59°$, $C=72°$, $a=200$
6. $A=84°$, $C=52°$, $c=160$
7. $A=66°$, $B=77°$, $a=801$
8. $A=10°$, $B=83°$, $c=100$
9. $B=81°$, $C=13°$, $c=6$
10. $A=65°$, $C=38°$, $b=3.84$

11. In an isosceles triangle, the base is 20 and a base angle is 50°. Find the legs.

12. In an isosceles triangle, the base is 12 and the vertex angle is 70°. Find the legs.

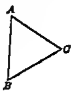
13. To find the distance from A to B across a river, an engineer measured a line AC and the $\angle A$ and C. If $AC=100$ ft., $A=86°$, and $C=67°$, what is the distance AB?

14. A balloon, C, is observed from two stations A and B, which are two miles apart, and on directly opposite sides of it. If the angle of elevation at A is 64°, and at B, 68°, how far is the balloon from A? Also, how high is the balloon above the earth?



15. From two points on the shore, A and B, 500 ft. apart, a ship C is observed. If $\angle A = 48°$, and $\angle B = 69°$, how far is the ship from B?

16. To locate an enemy field piece, the direction of the flash was noted from two points A and B. If $\angle A = 82°$, $\angle B = 47°$, and $AB = 3{,}000$ ft., how far was the gun from A?



17. In $\odot O$, chord AB equals 5 in., and $\widehat{AB}$ equals 88°. Find the radius of the circle.

18. In $\triangle ABC$, $\angle A = 70°$, $\angle B = 64°$, and $AB = 10$. Find AD, the bisector of $\angle A$.

A Self-Measuring Test

1. Define the *sine of an angle;* the *tangent of an angle.*

2. What relation of the sides to the functions of the opposite angles have you proved true in any triangle?

3. If a leg and the opposite acute angle of a right triangle are known, which function of that angle would you use to find the other leg? To find the hypotenuse?

4. If the hypotenuse and an acute angle of a right triangle are known, which function would you use to find a leg?

5. As an angle grows larger and approaches 90°, what value is its sine approaching? What is happening to its tangent?

6. Give two methods of proving triangles similar.

7. Give two proportions true for a right triangle when an altitude is drawn on its hypotenuse.

8. Give two propositions in which the product of two quantities equals the product of two other quantities.

0. Give the principal methods of proving:

a) Triangles congruent. (e) Angles equal.
b) Lines parallel. (f) Arcs equal.
c) Lines equal. (g) Chords equal.
d) Lines perpendicular. (h) A quadrilateral a parallelogram.

0. State, in terms of intercepted arcs, what these angles are asured by:

(a) Central angle.
(b) Inscribed angle.
(c) An angle whose vertex is inside the circle
(d) An angle whose vertex is on the circle.
(e) An angle whose vertex is outside the circle.

MMARY OF THE PRINCIPAL METHODS OF BOOK THREE

243. A. Triangles are similar, if:

1. Two angles of one triangle equal respectively two gles of the other (§ 216).
2. An angle of one triangle equals an angle of another, l the including sides are proportional (§ 220).
3. Right triangles, having an acute angle of one . . ., etc. 217).
4. They are similar to the same triangle (§ 218).

B. Lines are proportional, if they are:

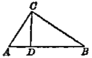
1. Corresponding sides of similar triangles (§§ 223, 215).
2 Segments of two sides of a triangle made by a line rallel to the third side (§ 208).

C. To prove two products equal, use the fact that:

1. In a proportion, the product of the extremes equals the oduct of the means (§ 200).
2. If two chords intersect, the product of the segments one . . . (§ 230).

D. Useful properties of the right triangle. In $\triangle ABC$, $\angle C$ is the right angle and CD perpendicular to AB.

1. $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$ (§ 227).
2. $\frac{DA}{DC} = \frac{DC}{DB}$ (§ 225).
3. $\frac{AD}{AC} = \frac{AC}{AB}$.
4. $\frac{BD}{BC} = \frac{BC}{BA}$.



(Note that all the terms of 2 begin with D, of 3 with A, and of 4 with B.)

Review Exercises; Honor Work

1. Find the product of the segments of a chord through a point 5 in. from the center of a circle whose radius is 9 in. Solve this problem by two different methods.

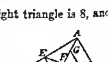
2. Find the diameter of the circle inscribed in a rhombus whose sides and one of whose diagonals are 10.

3. If the tangents at the ends of a chord 12 in. long meet at an angle of 60°, find the radius of the circle. Also find the length of the line segment from the point of intersection of the tangents to the center of the circle.

4. If a chord 8 in. long has an arc of 120°, find the radius.

5. If one angle of a right triangle is 30°, show that the altitude on the hypotenuse divides the hypotenuse into segments in the ratio of 3 to 1.

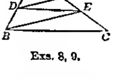
6. The altitude on the hypotenuse of a right triangle is 8, and it divides the hypotenuse into segments, one of which is four times the other. Find the hypotenuse.



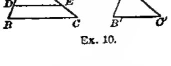
7. In $\triangle ABC$, AD and CE are medians, and G and H are the middle points of AF and FC respectively. Prove that GH is equal and parallel to ED.

8. If $DE \parallel BC$ and $DF \parallel BE$, prove $\frac{AF}{AE}=\frac{AE}{AC}$.

9. If $DE \parallel BC$ and $\frac{AF}{AE}=\frac{AE}{AC}$, then $DF \parallel BE$.



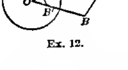
Exs. 8, 9.



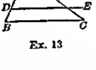
Ex. 10.

10. If, in $\triangle ABC$ and $A'B'C'$, $\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}$ and AD and AE equal respectively $A'B'$ and $A'C'$, prove that $DE=B'C'$. (*Use* § 202.)

11. Construct a circle through two given points tangent to a given line. (*Use* § 231.)



Ex. 12.

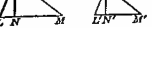


Ex. 13

12. If $\frac{OA}{r}=\frac{r}{OA'}$ and $\frac{OB}{r}=\frac{r}{OB'}$, where r is the radius of $\bigcirc O$, prove that $\triangle OAB \sim \triangle OA'B'$.

13. If $DE \parallel BC$ and $\frac{AB}{AD}=\frac{BC}{DE}$, then E is on AC.

14. If $\triangle KLM \sim \triangle K'L'M'$, $LN=\frac{1}{4}LM$, and $L'N'=\frac{1}{4}L'M'$, prove that $\angle KNL=\angle K'N'L'$.

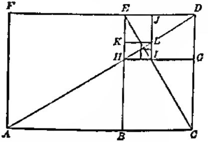


15. Similar triangles inscribed in equal circles are congruent.

16. Similar triangles are congruent, if a pair of corresponding altitudes are equal.

17. Similar quadrilaterals are cut into similar triangles by corresponding diagonals.

18. The side AB of square $ABEF$ is produced to C, so that $\frac{AC}{AB}=\frac{AB}{BC}$, and a rectangle $ACDF$ is constructed. Show that rectangle $BCDE$ is similar to rectangle $ACDF$.



19. In the same figure, prove that $\triangle ACD$ is similar to $\triangle CDE$.

20. In the same figure, prove that $CE \perp AD$

21. Also prove that $HB = BC$, and, if HG is drawn perpendicular to CD, that $BCGH$ is a square.

22. Then prove that $IG=GD$, and if $IJ \perp DE$, that $IGDJ$ is a square.

NOTE. This figure is known as the rectangle of whirling squares. It was used by the ancient Greeks as the basis of much of their art. Its proportions are considered by artists the most pleasing to the eye. The division of a line AC so that $\frac{AC}{AB}=\frac{AB}{BC}$ is called the golden section or golden mean.

23. If a picture frame must be 30 in. long, how wide should it be to conform to the Greek idea of proper proportions?

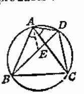
24. Parallelograms are similar, if an angle of one equals an angle of the other and the including sides are proportional.

25. If $\triangle ABC \sim \triangle A'B'C'$ and $\frac{AB}{A'B'}=\frac{BD}{B'D'}$, prove that $\triangle ADC \sim \triangle A'D'C'$.

26. Three lines, passing through a common point, cut off proportional segments on two parallel lines.

27. If two intersecting chords are perpendicular to each other, the sum of the squares of the four segments equals the square of the diameter.

28. In an inscribed quadrilateral, the sum of the products of the opposite sides equals the product of the diagonals. (Construct $\angle BAE = \angle CAD$. Then prove $\triangle ABE \sim \triangle ACD$ and $\triangle AED \sim \triangle ABC$.)



29. Corollary to Ex. 28. If $\triangle ABC$ is equilateral and D any point on $\widehat{AC}$, then $BD = AD + DC$.

30. If three circles intersect each other, the common chords meet in a point.

31. Tangents at the ends of a leg of an inscribed isosceles triangle form with the leg a triangle similar to the original triangle.

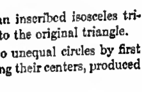
32. Construct a common tangent to two unequal circles by first finding the point where it cuts the line joining their centers, produced if necessary.

33. Construct $x = \frac{abc}{de}$, where a, b, c, d, and e are given segments.

34. On a given line segment as base, construct a triangle similar to a given triangle.

35. The middle points of the sides of a square are the vertices of a square.

36. If circles are circumscribed about similar triangles, their radii are proportional to the corresponding sides.



37. BC and BD are tangents to $\odot O$ at the ends of chord CD. Prove that $OA \times OB = \overline{OE}^2$.

38. OE equals 12. Find the length of OB when OA equals (a) 1; (b) 2; (c) 3; (d) 4; (e) 6; (f) 12.

39. As point A moves from the center of the circle toward the outer end of radius OE, (a) describe the motion of point B. When A reaches E, where is B? Where now are the points C and D?

(b) What change is taking place in the size of $\angle CBD$? When A reaches E, what size has $\angle CBD$? What has become of the two tangents CB and BD?

40. *Two triangles which have their sides respectively parallel or perpendicular are similar.*

In each pair of corresponding angles, the angles are either equal or supplementary (§ 84). Two pairs cannot be supplementary for the sum of all the angles would equal more than two straight angles. Therefore, two pairs are equal.

41. If the square of one side of a triangle equals the sum of the squares of the other two sides, the angle opposite the longest side is a right angle.

Construct a right triangle having its legs equal to the shorter two sides, and prove the triangles congruent by § 57.

BOOK FOUR

AREAS OF POLYGONS

244. The unit of surface is a square whose side is a unit of length.

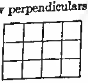
Examples of this are the square inch, the square yard, the square meter and the square mile.

245. The area of a surface is the number of units of surface it contains. It is the ratio of the surface to the unit of surface.

For convenience, it is customary to use the simple word triangle, rectangle, or circle, when the area contained within a triangle, rectangle, or circle, respectively, is meant. Thus we say, "Rectangles are to each other as, etc."

Exercises

1. Draw a rectangle 4 in. long and 3 in. wide. Divide the base and altitude into segments each 1 in. long and draw perpendiculars to the sides through the points of division, cutting the rectangle into squares 1 in. on a side. How many squares are there in one row the length of the rectangle? How many rows are there? Then how many squares are there in the rectangle and what is its area in square inches?

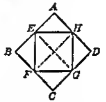


2. Into how many squares 1 in. on a side could you cut a rec-

tangle 5 in. long by 2 in. wide? 8 in. long by 3 in. wide? a in. long by b in. wide, if a and b are whole numbers? State a rule for finding the area of a rectangle which is true for these cases.

3. Draw a rectangle $2\frac{3}{4}$ in. long and $1\frac{1}{4}$ in. wide and cut it into squares $\frac{1}{4}$ in. on a side. How many squares are there in a row? How many rows are there? What is the total number of squares, and how many of them does it take to make a square 1 in. on a side? Then how many square inches are there in the area of the rectangle? Does the product of $2\frac{3}{4}$ and $1\frac{1}{4}$ give the correct area?

4. The middle points $E, F, G,$ and H of the sides of square $ABCD$ are joined in succession. Prove that $EFGH$ is a square. If AB is 6 units long, prove that $EF=3\sqrt{2}$.



5. Using the same hypothesis, prove that $EFGH$ is one half of $ABCD$.

6. If $AB=6$, find the area of $ABCD$, and from this the area of $EFGH$. If you multiply the dimensions EF by EH, that is, $3\sqrt{2}\times3\sqrt{2}$, do you obtain the same value?

7. If a rectangle has the following length and width, find a length which will be contained in both dimensions a whole number of times. Then, if the figure is divided, as in exercise 1, find the number of squares in a row and the number of rows, and from this the total number of squares and the area in square inches. (a) 2.2 by 1.3; (b) 2.17 by 1.32; (c) 2.174 by 1.323. In each of these cases, how does the result compare with the product of the length and width of the rectangle?

8. The base of a rectangle is $\sqrt{3}$ in. long and the altitude is $\sqrt{2}$ in. long. There is no unit which is contained in both of these lengths a whole number of times, so we shall take first .1 in., then .01 in., etc as a unit, and find the nearest whole number of times it is contained in the base and altitude. This will give an approximate value, for evidently the remainder will be smaller than the unit chosen. The work is shown in the following table. Complete the table and compare the sum of the areas of the squares with the value of $\sqrt{6}$, the product of the base and altitude.

Unit	Number of Times Contained in $\sqrt{2}=1.414$	Number of Times Contained in $\sqrt{3}=1.732$	Approximate Number of Squares	Number of Unit Squares to the Sq. In.	Approximate Area in Sq. In.	Value of $\sqrt{6}$
.1	14	17	240	100	2.4	
.01	141	173	24,400	10,000	2.44	
.001	1414	1732	2,449,000	1,000,000		

Does the result obtained from the small squares justify us in assuming that the area of a rectangle equals the product of its base and altitude?

246. Assumption. *The area of a rectangle equals the product of its base and altitude.*

247. Corollary 1. *Rectangles having equal altitudes are to each other as their bases.*

248. Corollary 2. *Rectangles having equal bases are to each other as their altitudes.*

249. Corollary 3. *Rectangles are to each other as the products of their bases and altitudes.*

Class Exercises

1. Find the area of a rectangle whose length is 276 and whose width is 92.

2. If two rectangles have the same base b, and their altitudes are respectively c and d, what is the ratio of their areas?

3. If the base of a rectangle is doubled while the altitude remains unchanged, what change takes place in the area?

4. If the numerical measure of a rectangle is 144 when a square 1 ft. on a side is the unit of area, what number would express its area, if the unit of area were a square 1 yd. on a side?

5. What is the altitude of a rectangle, if its area is A and its base is b?

6. How many bricks 4 in. by 6 in. will it take to pave a sidewalk 40 ft. long and 5 ft. wide?

7. If one rectangle is x ft. long and 6 ft. wide, and another rectangle is $x+5$ ft. long and 6 ft. wide, by how many square feet does the second rectangle exceed the first rectangle?

8. Find the side of a square equal in area to a rectangle, whose base is 24 and whose altitude is 6.

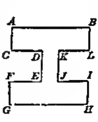
9. If the base of a rectangle is 12 and a diagonal is 13, find its area.

10. Find the area of an inscribed rectangle whose base is 10, if the radius of the circle is 13.

11. Find the area of a square inscribed in a circle whose radius is 8.

12. If the area of a square is 196, what is the radius of the circumscribed circle?

13. Find the area of the figure, if $AB=20$, $BL=AC=FG=6$, $CD=DE=EF=KL=KJ=JI=8$, and the angles are all right angles.

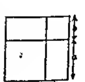


14. Divide a rectangle into three equal parts by lines parallel to one side.

15. Divide a rectangle into two parts in the ratio of m to n by a line parallel to one side.

Optional Exercises

16. Find the area of a square whose side is $a+b$. Find the area of each of the parts shown in the figure.



17. Show geometrically that $(a+b)^2=a^2+2ab+b^2$.

18. Show geometrically that $(a-b)^2=a^2-2ab+b^2$.

19. A rectangle whose base is 12 is inscribed in a circle whose radius is 10. How does its area compare with that of another rectangle, whose base is 16, inscribed in the same circle?

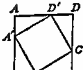
20. Construct a square equal to the sum of two squares, whose areas are 81 and 144 respectively.

21. A rectangle whose base is twice its altitude is inscribed in a circle whose radius is 10. Find the area of the rectangle.

22. A rectangle whose width is 6 equals a square whose side is 12. Find their perimeters.

Honor Work

23. Find the area of a square inscribed in a semicircle whose radius is 10.



24. In the square $ABCD$, $AA' = BB' = CC' = DD' = \frac{2}{3}AB$. Prove that $A'B'C'D'$ is a square, and find its area if $AB = s$.

25. If the altitude of a rectangle increases, how must the base change if the area is to remain unchanged? When the altitude has become three times its original length, how does the base compare with its original length?

26. State what change is taking place in the area of a rectangle, if: (a) its base is increasing while its altitude remains constant: (b) both base and altitude are increasing; (c) the base is increasing and the altitude decreasing. Discuss (c).

27. The base of a square increases 4 ft. while its altitude decreases 4 ft. Is the area increased or decreased, and by how many square feet?

Applied Problems

28. How many bricks will be needed to pave a sidewalk 50 ft. long and 6 ft. wide, if the bricks are 8 in. by 4 in.?

29. Find the number of sheets of tin 2 ft. by 18 in. for a flat roof 40 ft. long and 24 ft. wide.

30. Ordinary shingles are laid so that they average about 18 sq. in. exposed. If a building is 18 ft. long, 16 ft. wide and the ridge is 6 ft.

above the eaves, how many shingles will be needed? If they are sold in bunches of 250 each, how many bunches must be bought?

31. A woman wishes to buy linoleum to cover the kitchen floor. She measures the kitchen and finds that it is 15 ft. long and 11 ft. wide. How many square yards will she need?

32. What will it cost to plaster the walls and ceiling of the kitchen in Ex. 31 at $1 per sq. yd., if its height is $8\frac{1}{2}$ ft., and an allowance of 9 sq. yds. is made for doors and windows?

33. A woman finds that she can buy a rug for the living room for $65, or carpet at $3 a square yard. The living room is 12 ft. square. She buys the rug. How much could she have saved if she had taken the carpet?

34. A dress pattern calls for 3 yds. of cloth 1 yd. wide, but the material which Mrs. Brown wishes to buy is 42 in. wide. Assuming that she can use it to the same advantage, what length should she buy?

35. Dorothy is making paper dolls. She cuts each doll from a rectangle 5 in. by 3 in. How many dolls can she cut from a rectangle of paper 20 in. by 15 in.?

36. A farmer buys a field 64 rods by 50 rods at $25 an acre. What price does he pay? (An acre equals 160 square rods.)

37. Mr. Miner needs 600 sq. ft. of tin to cover the roof of his new house. What length must his plumber cut from a roll 3 ft. wide?

38. How many board feet (square feet) of lumber are needed for the floor of a room 11 ft. long and 9 ft. wide? If the boards are 3 in. wide, what is their total length?

39. Mrs. Lewis has a linen tablecloth 7 ft. by 4 ft. She wishes to make six napkins from a part of it, and still have a tablecloth 4 ft. square. How much cloth can she use for each napkin?

40. A room is 8 ft. high, and the distance around it is 45 ft. How many rolls of wall paper will be needed for the walls, if each roll is 18 in. wide and 48 ft. long? Make no allowance for doors or windows.

41. If a bolt of cloth is 28 in. wide, what length must be taken in order to obtain $3\frac{1}{2}$ sq. yds.?

Algebraic Exercises

42. The length of a rectangle is 10 ft. more than the width. If each dimension is increased 5 ft., the area will be increased 475 sq. ft. Find its dimensions.

43. The length of a rectangle is three times the width. If the length is decreased 4 and the width is increased 3, the area will be increased 8. Find its dimensions.

44. Find the side of a square whose area is increased 32 when the dimensions are each increased 2.

45. If the base of a square is increased 3 and the altitude decreased 2, the area remains unchanged. Find its side.

46. If the length of a rectangle is increased 4 and the width diminished 2, or, if the length is decreased 4 and the width increased 4, the area remains unchanged. Find the dimensions.

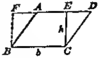
47. A square rug is 2 ft. from each wall in a room and covers one-half the floor. Find the length of the rug correct to the tenth of a foot.

48. A woman has the choice of two rugs, both having the same area, for her square dining room. One of them is the length of the room but 4 ft. narrower, whereas the other is 2 ft. shorter than the room and $2\frac{1}{2}$ ft. narrower. Find the dimensions of the room.

49. If I cut a strip an inch wide from all sides of a rectangular sheet of paper, I reduce its area by one-half, whereas, if instead I cut a strip the same width from the two ends only, the sheet becomes square. Find its dimensions.

50. Illustrate geometrically that:

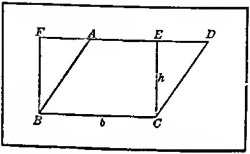
(a) $(a+b)(c+d)=ac+bc+ad+bd.$
(b) $(a+b)(c-d)=ac+bc-ad-bd.$
(c) $(a-b)(c-d)=ac-bc-ad+bd.$



Investigation Problem. How does the area of $\square ABCD$ compare with that of the rectangle $FBCE$? State and prove a proposition about the area of a parallelogram.

Proposition 1

* 250. *The area of a parallelogram equals the product of its base and altitude.*



Given: $\square ABCD$, with base $BC=b$, and altitude $CE=h$.
To prove: $ABCD=hb$.

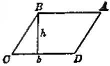
Proof:

Statements	Reasons
1. Extend DA and construct $BF \perp DA$ at F.	1. Post. 2 and § 64.
2. $FBCE$ is a rectangle.	2. § 101.
3. In $\triangle ABF$ and DCE, $AB=DC$ and $BF=CE$.	3. § 105.
4. $\angle F$ and $\angle CED$ are rt. $\angle s$.	4. § 12.
5. $\triangle ABF \cong \triangle DCE$.	5. § 94.
6. $ABCE \cong ABCE$.	6. Iden.
7. $ABF+ABCE=DCE+ABCE$, or $FBCE=ABCD$.	7. Ax. 3.
8. $FBCE=hb$.	8. § 246.
9. $ABCD=hb$.	9. Ax. 2.

251. Corollary 1. *Parallelograms having equal altitudes are to each other as their bases.*

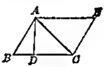
252. Corollary 2. *Parallelograms are to each other as the products of their bases and altitudes.*

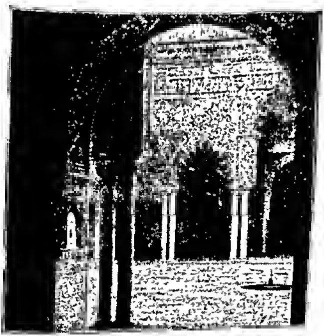
253.[1] **Corollary 3. Trigonometry.** *If two adjacent sides and the included angle of a parallelogram are a, b, and C, respectively, and the area is K, then* $K = ab \sin C$.



$$K = hb \text{ (§ 250)}, \quad h = a \sin C \text{ (§ 238)}, \quad K = ab \sin C$$

Investigation Problem. How does the area of $\triangle ABC$ compare with that of $\square ABCE$? State and prove a proposition about the area of a triangle.



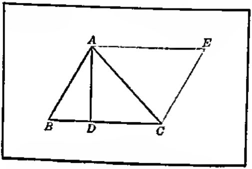


THE USE OF THE PARALLELOGRAM AND CIRCLE IN ARCHITECTURAL DESIGN

[1] Optional.

Proposition 2

254. *The area of a triangle equals half the product of its base and altitude.*



Given: $\triangle ABC$, with altitude AD.

To prove: $\triangle ABC = \frac{1}{2}AD \times BC$.

Proof:

Statements	Reason
1. Construct $CE \parallel AB$, and $AE \parallel BC$, meeting at E.	1. § 77.
2. $ABCE$ is a $\square$.	2. § 100.
3. $\square ABCE = AD \times BC$.	3. § 250.
4. $\triangle ABC = \frac{1}{2}\square ABCE$.	4. § 104.
5. $\triangle ABC = \frac{1}{2}AD \times BC$.	5. Ax. 6.

255. **Corollary 1.** *Triangles having equal altitudes are to each other as their bases.*

256. **Corollary 2.** *Triangles having equal bases and equal altitudes have equal areas.*

257. **Corollary 3.** *Triangles are to each other as the products of their bases and altitudes.*

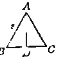
258.[1] **Corollary 4. Trigonometry.** *The area of a triangle equals one-half the product of two sides and the sine of their included angle.*

$$K = \tfrac{1}{2}ab \sin C.$$

[1] Optional.

259.[1] Corollary 5. *In an equilateral triangle, whose side is s, the altitude is* $\frac{s}{2}\sqrt{3}$ *and the area is* $\frac{s^2}{4}\sqrt{3}$.

Show that: $BD = \frac{s}{2}$ (§ 95). $AD = \frac{s}{2}\sqrt{3}$ (§ 228).



$$\triangle ABC = \frac{s^2}{4}\sqrt{3} \text{ (§ 254).}$$

CLASS EXERCISES

1. Find the area of a triangle whose base is 12 and whose altitude is 9.

2. Find the area of a right triangle whose legs are 8 and 9.

3. Find the area of a right triangle whose hypotenuse is 13 and one of whose legs is 12.

4. Find the area of a parallelogram, whose base is 19 and whose altitude is 12.

5. If one parallelogram has half the base and half the altitude of another, it equals one-fourth of the other.

6. A triangle having half the base and half the altitude of a parallelogram, is one-eighth of the parallelogram.

7. The area of a rhombus equals one-half the product of its diagonals.

8. Find the area of a rhombus, the diagonals of which are 36 and 40.

9. Find the area of a rhombus, if a side is 10 and a diagonal 12.

10. Find the area of a parallelogram, if two sides are 16 and 22 and an angle is 30°.

11. Find the area of a rhombus, if the perimeter is 52 and one diagonal is 10.

12. If a triangle and a parallelogram on the same base are equal in area, compare their altitudes.

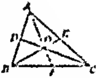
13. The line joining the middle points of two adjacent sides of a parallelogram cuts off one-eighth of the parallelogram.

14. A median of a triangle bisects the triangle.

[1] Optional.

OPTIONAL EXERCISES

15. If D and E are the middle points of the sides AB and AC respectively of $\triangle ABC$, prove that $\triangle BDC$ has the same area as $\triangle BEC$.



16. Using the same hypothesis, prove that $\triangle BOD$ is equal in area to $\triangle COE$.

17. If the medians of a triangle meet in a point, they form six triangles, all equal in area.

18. If AF, BE, and CD are medians of $\triangle ABC$, then $\triangle AOC = 2\triangle COF$.

19. Using the same hypothesis, $AO = 2 \cdot OF$, or OF is one-third of AF.

20. If one diagonal of a quadrilateral bisects the other diagonal, it bisects the quadrilateral.

21. The diagonals of a parallelogram divide the figure into four triangles, all equal in area.

22. The base of a triangle is 12. Find its altitude, if its area equals that of a parallelogram whose base is 8 and whose altitude is 9.

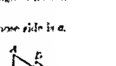
23. Find the area of an equilateral triangle whose side is 10.

24. Find the area of an equilateral triangle whose altitude is $3\sqrt{3}$.

25. Find the altitude of an equilateral triangle whose area is $25\sqrt{3}$.

26. Find the area of an equilateral triangle whose side is a.

27. If $AB \perp BC$, but DB is not perpendicular to BC, and $AB = DB$, prove that $\triangle ABC$ is greater than $\triangle DBC$.



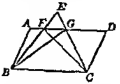
28. Find the area of an isosceles right triangle if the altitude on the hypotenuse is 6.

29. Find the area of an isosceles right triangle if the hypotenuse is 10.

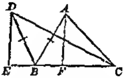
30. The square on the hypotenuse of an isosceles right triangle is four times the triangle.

31. The square on the diagonal of a square is double the original square.

32. If $ABCD$ is a parallelogram, prove that $\triangle EBG$ is equal in area to $\triangle EFC$.



Ex. 32.



Ex. 34.

33. Find the area of a triangle whose base is 10 and whose base angles are 120° and 30° respectively.

34. Two triangles are equal in area, if two sides of one equal two sides of the other, and the included angles are supplementary.

Honor Work

35. If E is any point inside $\square ABCD$, the sum of $\triangle EAD$ and EBC equals one-half the parallelogram.

36. The sum of the perpendiculars from any point in the base of an isosceles triangle on the legs equals the altitude on a leg.

37. The sum of the perpendiculars from any point inside an equilateral triangle on the three sides equals the altitude.

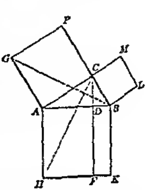
38. The area of a triangle equals one-half the product of the perimeter and the radius of the inscribed circle.

39. The area of a circumscribed polygon equals one-half the product of the perimeter and the radius of the inscribed circle.

40. Find the area of an equilateral triangle inscribed in a circle whose radius is 10.

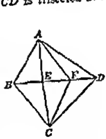
41. Prove, without using the algebraic method of Book Three, *that the square on the hypotenuse of a right triangle equals the sum of the squares on the legs.*

SUGGESTION. Construct $CDF \perp AB$, and draw BG and CH. The plan is to prove $\triangle BGA$ congruent to $\triangle CAH$, and then to prove these triangles half the square $PGAC$ and the rectangle $DAHF$ respectively. Try to prove it without further help. Is PCB a straight line? If AG is taken as the base of $\triangle BGA$, what is its altitude? How does its area compare with that of square $PGAC$? How does $\triangle CAH$ compare in area with rectangle $AHFD$? Now compare $PGAC$ with $AHFD$. Compare $MCBL$ with $DFKB$.



42. If the diagonal AC of quadrilateral $ABCD$ divides the figure into two equal areas, it bisects BD.

43. The diagonal BD of quadrilateral $ABCD$ is trisected at E and F. Prove that the quadrilaterals $ABCE$, $AECF$, and $AFCD$ are equal in area.



44. If side DC of rectangle $ABCD$ is extended to E, prove that $\triangle ABE$ equals one-half rectangle $ABCD$.

45. Two isosceles triangles have the same area, if their legs are equal and the base of one is twice the altitude of the other.

46. Any line through the point of intersection of the diagonals of a parallelogram bisects the parallelogram.

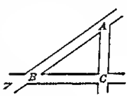
47. If the sides of a parallelogram remain constant, but an angle increases from 0° to 90°, how does the area change? If two adjacent sides are 10 and 12, find the area of the parallelogram, when the included angle is 30°; 45°; 60°; and 90°.

48. *Trigonometry.* If the sides of a parallelogram remain constant, but the included angle changes, prove that the area varies as the sine of that angle.

Applied Problems

49. A house whose width is 30 ft., has two gabled ends, and the ridge is 20 ft. above the plate at the eaves. How many square feet of boards are needed to cover both gables?

50. Two streets AB and BC meet at an angle of 30°, and a third street AC crosses BC at right angles. If BC is 80 yds., find the area of the plot bounded by the three streets.



51. A man builds a corner seat in the form of a right triangle whose legs are 3 ft. and 5 ft. If he wishes to cover it with leather, how many square feet will he need?

52. South America is roughly a triangle whose base, running north and south, is 5,000 miles, and whose altitude, running east and west, is 3,000 miles. What is its approximate area in square miles?

53. In the year 1609, Kepler discovered that the speed at which the earth travels around the sun changes when its distance from the sun changes, and that, if he took the distance from the sun to the earth as the altitude of a triangle and the distance the earth traveled in a second as its base, the triangles always had the same area. If the earth goes 19 mi. per second when it is 92,000,000 miles from the sun, how fast is it traveling when it is 95,000,000 miles from the sun? What is the distance from the earth to the sun when the speed is $18\frac{1}{2}$ mi. per second?

54. As the earth moves farther from the sun, does its speed increase or decrease?

Geometric reasoning applied to life situations (Optional)

1. Assuming that it rains whenever the wind is in the east, which of the following statements are necessarily true?

   a. Since it is raining, the wind is in the east.
   b. Since the wind is in the east, it is raining.
   c. Since it is not raining, the wind is not in the east.
   d. Since the wind is not in the east, it is not raining.

2. If it is true that it rains only when the wind is in the east, which of the above statements are necessarily true?

BUILDING A SKYSCRAPER

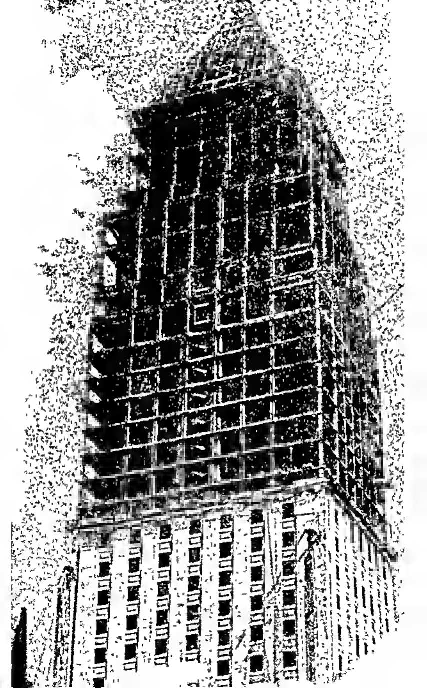
Large buildings are completely planned before their construction is started. Some one must compute the amount of surface to be enclosed, the areas of floors, ceilings, and walls, and all this must be done accurately from the plans before the contractor can bid for the contract. If the bid is too high, he will not get the work; if too low, he will lose money on it.

After the building is up, it is still necessary to plaster, paint, put on roofing, put in windows and rugs or carpeting, and perform numerous other operations that require finding areas. This work is usually let to other contractors whose office forces figure the entire cost before a contract is taken.

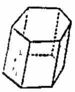
And after all this work is completed, there are still many places in the home in which the thrifty housewife can save money if she knows how to find areas of the materials needed for curtains, decorations, dresses, and other articles she must make.

As in the building shown here, the contractor does not necessarily begin walling in the rooms from the bottom up, but may for special reasons enclose rooms on certain floors, leaving those above and below open until later. He can do this because the walls are laid on the steel frame and not built up from the ground.

Photograph by George A. Douglas from Philip D. Gendreau, N. Y.



SPACE GEOMETRY (*Optional*)



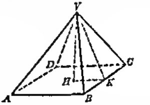
1. A prism is called a **right prism** if its lateral edges are perpendicular to its bases. Find the lateral area and also the total area of a right prism whose lateral edge is 10 in. and whose base is:

   (*a*) A square 6 in. on a side.
   (*b*) An equilateral triangle 5 in. on a side.
   (*c*) A rectangle 7 in. long and 4 in. wide.
   (*d*) A right triangle whose legs are 6 and 8 in.

2. How much cardboard is required to make a rectangular box without a cover, 6 in. long, 5 in. wide, and 4 in. deep?

3. Make a formula for the total area of a rectangular box whose length is l, width w, and height h.

4. A room is 12 ft. long, 10 ft. wide, and 9 ft. high. If we allow 70 sq. ft. for doors and windows, how many square feet of plaster are needed for the walls and ceiling?



In the pyramid shown here, VH is the altitude, H the middle point of the square base, and HK and VK are $\perp BC$. VK, the altitude of a triangle, is called the **slant height** of the pyramid.

5. Find the total surface of a square pyramid if each side of the base is 10 and the slant height is 12.

6. Find the lateral surface of a square pyramid if each side of the base is 6 and the altitude VH is 4.

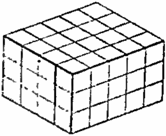
7. Make a formula for the lateral area L of a pyramid if the base has n sides, each of length e, and all lateral triangles have the same slant height s.

8. Prove that the lateral area of this pyramid equals half the product of the slant height and the perimeter of the base.

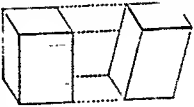
9. If the lateral area of this square pyramid is 260 and the perimeter of its base is 40, find its slant height, the edge of its base, and its altitude.

Volume of the rectangular box.

Just as we found by approximating that the area of the rectangle equalled the product of its two dimensions, so we could show that the volume of the rectangular box equals the product of its three dimensions, or its base multiplied by its altitude. In the illustration each layer contains 5×4 cubes and there are 3 layers, therefore 5×4×3 cubes.



If we regard the rectangular box as a pile of cards, we see that we could push it sidewise into the form of an oblique parallelepiped without changing the base, altitude, or volume. Consequently the volume of any parallelepiped equals its base times its altitude.



EXERCISES

1. How many pupils should occupy a classroom 20 ft. long, 14 ft. wide, and 10 ft. high if there should be allowed 80 cu. ft. of space for each pupil?

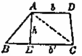
2. How many bushels will a bin 5 ft. by 3 ft. by 4 ft. hold if 1 bu. = 2150 cu. in.?

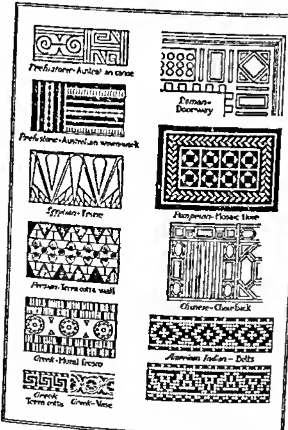
3. How many tons of coal will a bin 12 ft. long, 6 ft. wide, and 3 ft. high hold if a ton of coal occupies 36 cu. ft.?

4. A cord of wood is 8 ft. long, 4 ft. wide, and 4 ft. high. How many cords are there in a pile 20 ft. long, 8 ft. wide, and 6 ft. high?

5. Mrs. Brown ordered 25 lbs. of ice and the iceman brought her a piece 9 in. long, 8 in. wide, and 8 in. high. If a cu. ft. of ice weighs 57 lbs., did she receive 25 lbs.?

Investigation Problem. $ABCD$ is a trapezoid, AD being parallel to BC. What is the area of $\triangle ABC$? Of $\triangle ADC$? If BC and AD are respectively the bases of these triangles, what are their altitudes? How do they compare in length? Can you now state and prove a proposition about the area of a trapezoid?

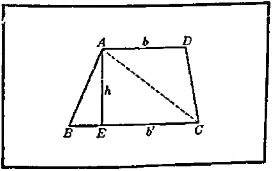




How mankind has used geometric figures in design

Proposition 3

* 260. *The area of a trapezoid equals half its altitude times the sum of its bases.*



Given: Trapezoid $ABCD$, with altitude h, and bases b and b'.

To prove: $ABCD = \frac{1}{2}h(b+b')$.

Proof:

Statements	Reasons
1. Draw AC.	1. Post. 1.
2. h is the altitude of both $\triangle$.	2. § 106.
3. $\triangle ABC = \frac{1}{2}hb'$ and $\triangle ACD = \frac{1}{2}hb$.	3. § 254.
4. $ABCD = \frac{1}{2}h(b+b')$.	4. Ax. 3.

Class Exercises

1. Find the area of a trapezoid whose bases are 6 and 14 and whose altitude is 11.

2. Find the area of a trapezoid whose bases are 6.83 and 3.17 and whose altitude is 17.

3. Find the area of trapezoid $ABCD$, if its bases AD and BC are 8 and 10 respectively, and if AB is 14, and $\angle B$ is 30°. Also if $\angle B$ is 45°.

4. Find the altitude of a trapezoid whose area is 144 and whose bases are 10 and 14.

5. If a line is drawn parallel to one leg of a trapezoid, and bisecting the other leg, the parallelogram formed by extending the shorter base is equal in area to the trapezoid.

6. From a rectangular piece of cardboard 12 in.$\times$14 in., John cut an isosceles trapezoid with bases 14 in. and 8 in. and legs 5 in., and a square with side 8 in. Find the area of the part remaining.

7. The bases of a trapezoid are 12 and 18 and the legs are each 6. Find the area.

8. The area of a trapezoid is 204 sq. in. Its altitude is 17 in. and one base is 16 in. Find the other base.

9. Find the side of a square equal to a trapezoid whose bases are 56 and 44 and whose legs are each 10.

Optional Exercises

10. Find the area of a trapezoid whose bases are 12 and 20 and whose base angles are each 60°.

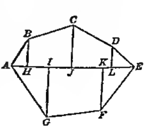
11. A trapezoid contains 240 sq. ft., and its altitude is 16 ft. Find the bases if one is 3 ft. longer than the other.

12. If the base of a triangle is 10 in. and its altitude 4 in., find the area of a trapezoid cut off by a line 3 in. from the vertex.

13. If E is the middle point of CD, one of the non-parallel sides of trapezoid $ABCD$, prove that the area of $\triangle ABE$ is one-half the area of the trapezoid.

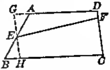
14. The line joining the middle points of the bases of a trapezoid bisects the area of the trapezoid.

15. To find the area of polygon $ABCDEFG$ the diagonal AE is drawn, and perpendiculars to it are constructed from B, C, D, F, and G. If $AH=5$, $HI=6$, $IJ=6$, $JK=9$, $KL=2$, $LE=7$, $BH=8$, $CJ=12$, $DL=6$, $KF=14$, and $IG=15$, find the area of $ABCDEFG$.



Honor Work

16. The area of a trapezoid equals the product of one leg and a perpendicular drawn to it from the middle point of the other leg.



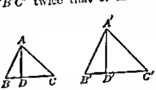
17. If the longer base of a trapezoid grows shorter until it is the same length as the other base, what has the trapezoid become? Does the formula for the area of a trapezoid still hold true for this figure?

18. If the shorter base of a trapezoid grows shorter until its length becomes zero, what has the trapezoid become? Does the formula for the area of a trapezoid still hold true for this figure?

19. Find the area of trapezoid $ABCD$, if $BC=20$, $CD=8$, $\angle A=90°$, $\angle B=90°$, and $\angle C=45°$.

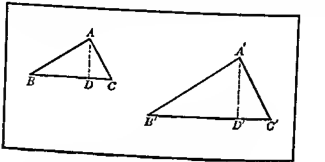
20. Find the area of a trapezoid if $AB=10$, $BC=24$, $\angle A=120°$, $\angle B=60°$, and $\angle C=90°$.

Investigation Problem. Draw a $\triangle ABC$. Now draw another $\triangle A'B'C'$ whose sides are each twice as long as the corresponding sides of $\triangle ABC$. Is the area of $\triangle A'B'C'$ twice that of $\triangle ABC$? More than twice? Cut $\triangle A'B'C'$ into triangles having sides equal to those of $\triangle ABC$. How many such triangles are there? If the sides of $\triangle A'B'C'$ were three times the corresponding sides of $\triangle ABC$, how would their areas compare? Test your conclusion by cutting $\triangle A'B'C'$ into triangles having sides equal to those of $\triangle ABC$. How then do the areas of similar triangles compare? Can you prove this? If not, answer the following questions. In the figure, what is the area of $\triangle ABC$? Of $\triangle A'B'C'$? What is the ratio $\dfrac{\triangle ABC}{\triangle A'B'C'}$? How does $\dfrac{AD}{A'D'}$ compare with $\dfrac{BC}{B'C'}$? Can you now find the ratio in terms of $\dfrac{BC}{B'C'}$? State your result as a proposition.



Proposition 4

* 261. *The areas of similar triangles are to each other as the squares of corresponding sides.*



Given: $\triangle ABC \sim \triangle A'B'C'$.

To prove: $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{\overline{BC}^2}{\overline{B'C'}^2} = \dfrac{\overline{AC}^2}{\overline{A'C'}^2} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.

Proof:

Statements	Reasons
1. Draw altitudes AD and $A'D'$.	1. § 64.
2. $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{BC}{B'C'} \times \dfrac{AD}{A'D'}$.	2. § 257.
3. $\dfrac{AD}{A'D'} = \dfrac{BC}{B'C'}$.	3. § 219.
4. $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{\overline{BC}^2}{\overline{B'C'}^2}$.	4. Ax. 1.
In the same way, $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{\overline{BC}^2}{\overline{B'C'}^2} = \dfrac{\overline{AC}^2}{\overline{A'C'}^2} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.	

Class Exercises

1. The sides of a triangle are 4, 5, and 6. Find the sides of a similar triangle nine times as large.

2. The bases of two similar triangles are 5 and 15. How do their areas compare?

3. The sides of a triangle are 5, 7, and 8. The perimeter of a similar triangle is 40. Compare their areas.

4. If the areas of two similar triangles are to each other as 2 is to 3, what is the ratio of their sides?

5. The sides of a triangle are 4, 5, and 7. Find the sides of a similar triangle whose area is three times as great.

6. Similar triangles are to each other as the squares of corresponding altitudes.

7. A park is represented on a map by a triangle whose area is 9 sq. in. If the scale of the map is 1 : 12,000, find the area of the park in square rods.

8. The base of a triangle is 8. Find the base of a similar triangle whose area is $2\frac{1}{4}$ times that of the former.

9. The base of a triangle is 15 and its area is 60. Find the area of a similar triangle whose altitude is 6.

10. The area of a right triangle is 25 and one leg is 10. Find the hypotenuse of a similar triangle whose area is 100.

Optional Exercises

11. The corresponding sides of two similar triangles are to each other as 3 is to 8, and the sum of their areas is 365. Find their areas.

12. What part of a triangle is cut off by a line parallel to the base and bisecting one side?

13. Two lines parallel to one side of a triangle trisect a second side. Compare the areas of the parts into which the triangle is cut.

14. The base of a triangle is 10 and its altitude 4. Find the area of the triangle cut off by a line parallel to the base and at a distance of 3 from the vertex.

15. The sides of a triangle are 6, 8, and 10. If a line parallel to the longest side bisects the area, find the segments into which it cuts the other two sides.

16. Construct a square twice as large as a given square.

17. Construct an equilateral triangle three times as large as a given equilateral triangle.

Honor Work

18. The diagonal of a rectangle is 13. The area of a similar rectangle is 240, and one of its sides is 24. What is the area of the first rectangle?

19. The base of a triangle is 16 in. and its altitude is 10 in. Find the area of the trapezoid cut off by a line 4 in. from the vertex.

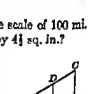
20. If equilateral triangles are constructed on the three sides of a right triangle, prove that the triangle on the hypotenuse equals the sum of the other two triangles.

Applied Problems

21. A sign painter has a picture 4 in. by 3 in., divided into right isosceles triangles for convenience, which he is to enlarge to form a sign 8 ft. in length. Find the width and the area of the sign. If each leg of each triangle is 1 in., what is the area? What area does it represent on the sign?

22. A map of the United States is drawn to the scale of 100 mi. to 1 in. What is the area of a state represented by $4\frac{1}{2}$ sq. in.?

23. A real-estate agent sells half of a triangular plot for a park, which is also to be triangular, and the other half for a building. He wishes to divide the plot by a line ED parallel to street BC. If AB is 100 ft., find the distance AE.



24. If a stone is dropped from a height, the speed at which it is falling at the end of a number of seconds can be found from the graph shown on page 333 where AC is a straight line. Find the speed at the end of 3 seconds; 6 seconds; 9 seconds.

25. Prove from the figure that the speed is proportional to the time.

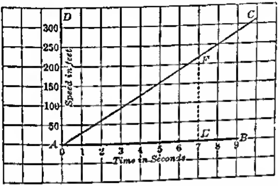
26. The distance fallen in any number of seconds is equal to the area of the triangle bounded by AB, AC, and the vertical line EF, where point E is on the horizontal line, at the number of seconds. Find the distance fallen in 3 seconds. In 6 seconds. In 9 seconds.

27. Prove from the figure that the distance fallen is proportional to the square of the time.

28. To find the height of a cliff, Paul dropped a stone and noted it required 4 seconds to reach the bottom. How high was the cliff?

29. If the stone were dropped from the top of a tower 400 ft. high, how many seconds would it take to reach the ground?

30. Using information from the graph, write a formula for finding the distance a stone will fall in any number of seconds.



Geometric reasoning applied to life situations (Optional).

1. Assume (*a*) that people on the average will spend one-fifth of their incomes for clothing, and (*b*) that machines are invented so that each man can make in a given time as much clothing as two men formerly did, but (*c*) that this reduces the price of clothing to half what it formerly cost. Will the use of these machines cause unemployment?

2. Mr. Brown spends part of his salary for necessities and the rest of it for luxuries. Mr. Rich, whose salary is twice Mr. Brown's, spends the same amount for necessities as Mr. Brown does, and spends the rest of his salary for luxuries. If the income tax ought to tax the part spent on luxuries, should Mr. Rich pay twice, more than twice, or less than twice as much tax as Mr. Brown?

3. Assume that reducing the gold content of the dollar will ultimately cause all wages and all costs of commodities and property to become double what they were before. Explain whether such reduction is an advantage or disadvantage to (*a*) The man who spends all of his wages each week; (*b*) The man who owns a house

on which there is a mortgage; (c) The man who holds the mortgage.

Completion Test (*10 min.*)

1. The area of a trapezoid whose bases are 8 and 10 and whose altitude is 9 is . . . .

2. The bases of a trapezoid are 6 and 8 and its area is 28. Its altitude is . . . .

3. The altitude of a parallelogram is a and its base is b. The side of an equal square is s. The proportion expressing the relation between a, b and s is . . . .

4. If the base of a triangle is doubled while the area remains unchanged, the altitude is multiplied by . . . .

5. If two adjacent sides of a parallelogram remain constant as the included angle increases to 90°, the area of the parallelogram must . . . .

Numerical Test (*10 min.*)

1. The base of a triangle is 8 and the altitude on it is 6. If the altitude on another side is 4, find the length of the side.

2. In the $\triangle ABC$ and DEF, $\angle A = \angle D$ and $\angle B = \angle E$. If $AB = 5$, $DE = 15$, and the area of $\triangle ABC$ is 14, find the area of $\triangle DEF$.

3. The legs of a right triangle are 3 and 4. Find the area of an equilateral triangle whose side equals the hypotenuse of the right triangle.

4. The lengths of the sides of a triangle are 4, 5 and 6. The perimeter of a similar triangle is 30. What is the ratio of the area of the second triangle to that of the first?

5. Corresponding sides of two similar triangles are in the ratio 1 : 4. Find the ratio of their perimeters.

Geometric designs based on the octagon.

Matching Test (*10 min.*)

After each number, write the letter of the phrase that forms a correct statement with that following the number.

1. Inscribed angle
2. Area of trapezoid
3. Triangles having equal altitudes
4. Areas of similar triangles
5. An angle formed by two tangents
6. Area of a triangle
7. Perimeters of similar triangles
8. Angle formed by intersecting chords
9. Unit of area
10. Area of a parallelogram

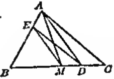
a. $\frac{1}{2}$ the altitude times the sum of the bases.
b. $\frac{1}{2}$ the product of base and altitude.
c. Product of base and altitude.
d. Are to each other as their bases.
e. Are to each other as the squares of corresponding sides.
f. A square whose side is a unit of length.
g. Are to each other as corresponding sides.
h. Measured by $\frac{1}{2}$ its arc.
i. Measured by half the sum of its arcs.
j. Measured by half the difference of its arcs.

Reasoning Test (*10 min.*)

Give the reason for each statement in the following proof:

Given: D any point on BC, AM the median to BC and $ME \parallel AD$.

To prove: $\triangle EBD = \frac{1}{2}\triangle ABC$.



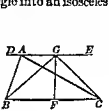
1. $BM = \frac{1}{2}BC$.
2. $\triangle ABM$ and ABC have the same altitude.
3. $\triangle ABM = \frac{1}{2}\triangle ABC$.
4. $ME = ME$.
5. $AD \parallel ME$.
6. The distance from A to ME = the distance from D to ME.
7. $\triangle AME = \triangle DME$.
8. $\triangle BME = \triangle BME$.
9. $\triangle ABM = \triangle EBD$.
10. $\triangle EBD = \frac{1}{2}\triangle ABC$.

262. Triangles on the same base are equal in area, if their vertices are on a line parallel to the base (§256).

ILLUSTRATION. Transform a given triangle into an isosceles triangle on the same base.

Given $\triangle ABC$.

To construct an isosceles triangle on BC equal in area to $\triangle ABC$.



Construct DE through A parallel to BC.

Construct FG, the perpendicular bisector of BC.

Draw GB and GC. Then $\triangle GBC$ is the required triangle.

EXERCISES

1. Transform a given $\triangle ABC$ into a right triangle having a leg equal to BC.

2. Transform a given $\triangle ABC$ into another triangle on BC, and having a given angle adjacent to BC.

3. Transform a given $\triangle ABC$ into a right triangle, having BC the hypotenuse.

4. Transform a given parallelogram into a rectangle on the same base.

5. Transform a given parallelogram into a parallelogram on the same base, and having a $60°$ angle.

6. Transform a given parallelogram into a rhombus on the same base.

7. Transform a given triangle into a triangle on the same base and having a side parallel to a given line.

8. Transform a given $\triangle ABC$ into an isosceles triangle having BC as a leg.

9. Transform a given triangle into another triangle on the same base, and having another side equal to a given line.

10. Transform a given triangle into another triangle having two given sides. (Make two transformations, one side at a time.)

11. Transform a triangle into another triangle having one given side, and a given angle adjacent to it.

A SELF-MEASURING TEST

1. Write a formula for the area of a rectangle; a triangle; a parallelogram; a trapezoid.

2. What are the principal propositions used in proving the one about the area of a trapezoid? a triangle? a parallelogram?

3. Complete this statement. Similar triangles are to each other as ......

4. Give the theorem in which parallelograms are to each other as their bases.

5. Are similar triangles to each other as the products of their bases and altitudes? Is this true if the triangles are not similar?

6. State the conditions under which triangles are to each other: (a) as their bases; (b) as the squares of their bases; (c) as their altitudes.

7. How would you find the area of an irregular polygon?

8. Define:

(a) *Quadrilateral.*
(b) *Parallelogram.*
(c) *Rhombus.*
(d) *Rectangle.*
(e) *Trapezoid.*
(f) *Vertical angles.*
(g) *Adjacent angles.*
(h) *Supplementary angles.*
(i) *Right angles.*
(j) *Isosceles triangle.*
(k) *Hypothesis.*
(l) *Hypotenuse.*
(m) *Indirect proof.*
(n) *Tangent.*
(o) *Chord.*
(p) *Inscribed angle.*
(q) *Arc.*
(r) *Similar triangles.*
(s) *Perpendicular.*
(t) *Extremes.*

9. What construction lines would you draw in order to prove that:

(a) Base angles of an isosceles triangle are equal?
(b) The sum of the angles of a triangle equals a straight angle?
(c) The area of a triangle equals . . .?
(d) A line which divides two sides of a triangle proportionally, . . .?
(e) An inscribed angle is measured by . . .?

(*f*) The area of a trapezoid equals . . .?

(*g*) The sum of the squares of the legs of a right triangle equals . . .?

(*h*) The product of the segments of a chord . . .?

(*i*) The angle formed by a tangent and chord is measured by . . .?

(*j*) Opposite sides of a parallelogram are equal?

(*k*) Lines are parallel if their alternate interior angles are equal?

(*l*) In a circle equal chords have equal arcs?

10. State the principal proposition on which each of these constructions depends·

(*a*) A line through a point parallel to a line.

(*b*) Dividing a line into three equal parts.

(*c*) Constructing a tangent at a point on a circle.

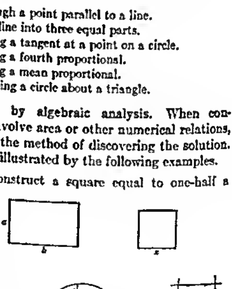
(*d*) Constructing a fourth proportional.

(*e*) Constructing a mean proportional.

(*f*) Circumscribing a circle about a triangle.

263. Construction by algebraic analysis. When construction exercises involve area or other numerical relations, algebraic analysis is the method of discovering the solution. This method will be illustrated by the following examples.

Illustration 1. Construct a square equal to one-half a given rectangle.



STEP 1. Draw a square to represent the one to be constructed.

STEP 2. We could construct the square if we knew a side. Therefore let x equal a side.

STEP 3. The useful known lines are the altitude and base of the given rectangle. Call them a and b respectively.

STEP 4. The area of the square is one-half that of the rectangle Therefore $x^2 = \frac{1}{2}ab$.

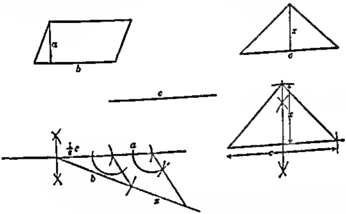
STEP 5. This equation can be transformed into the proportion $\frac{a}{x} = \frac{x}{\frac{1}{2}b}$.

STEP 6. Construct x. Bisect b and construct the mean proportional between a and $\frac{1}{2}b$.

STEP 7. Construct a square on x as a side.

Illustration 2. On a given line segment as base, construct an isosceles triangle equal in area to a given parallelogram.

STEP 1. Draw an isosceles triangle to represent the one to be constructed.



STEP 2. We could construct the triangle if we knew the altitude Therefore let x equal the altitude.

STEP 3. The useful known lines are the altitude and base of the given parallelogram and the given line segment. Call them a, b, and c, respectively.

STEP 4. The equality of the two areas gives us the equation $\frac{1}{2}cx = ab$.

STEP 5. This equation can be transformed into the proportion $\frac{\frac{1}{2}c}{a} = \frac{b}{x}$.

STEP 6. Construct x, the fourth proportional to $\frac{1}{2}c$, a, and b.

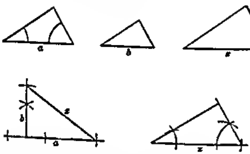
STEP 7. With c as base and x as altitude, construct the required triangle.

Illustration 3. Construct a triangle similar to two given similar triangles and equal to their sum.

STEP 1. Draw a triangle to represent the one to be constructed.

STEP 2. We could construct this triangle if we knew its base. Therefore let x equal its base.

STEP 3. The useful known lines are the bases of the given triangles. Call them a and b respectively.



STEP 4. From the condition given and § 261, $x^2=a^2+b^2$.

STEP 5. This equation is already in the form of the sum of two squares.

STEP 6. Construct x by § 227.

STEP 7. On x, construct a triangle similar to one of the given triangles.

From these illustrations, we observe that the method of discovering the solution of a construction by algebraic analysis is as follows:

1. Draw a figure to represent the one to be constructed.
2. Let x stand for a line, which, if known, would enable us to construct the figure.
3. Let a, b, c, etc., stand for useful given lines.

4. From the conditions given in the problem, write an equation in x, a, b, etc.

5. Transform this equation into a proportion or the sum of squares.

6. Construct x by § 213, § 229, or § 227.

7. Using this value of x, construct the required figure.

Class Exercises

1. Construct a square equal in area to:

(a) A given parallelogram.
(b) A given triangle.
(c) The sum of two given squares.
(d) Three times a given square.
(e) One-half a given square.
(f) The difference of two given squares.
(g) Twice a given triangle.

2. Construct a triangle similar to a given triangle, and having:

(a) Twice the area.
(b) One-third the area.
(c) Its altitude equal to a given line segment.

3. Bisect a triangle by drawing a line parallel to one of its sides.

4. Construct an isosceles triangle on a given line segment as base and equal in area to a given triangle.

5. Construct a right triangle equal in area to a given triangle, and having one leg equal to a given line segment.

6. Construct a rectangle, having its base equal to a given line segment:

(a) Equal in area to a given rectangle.
(b) Equal in area to a given triangle.
(c) Equal in area to a given square.
(d) Having twice the area of a given rectangle.
(e) Equal in area to a given rhombus.

7. Construct an equilateral triangle equal in area to the sum of two given equilateral triangles.

8. Construct an equilateral triangle equal in area to the difference of two given equilateral triangles.

9. Construct a square equal to the sum of three given squares.

Honor Work

10. Construct a rectangle on a given base and equal in area to a given trapezoid.

11. Construct a square equal in area to a given trapezoid.

12. Construct a parallelogram equal in area to a given parallelogram and having a given side and a given angle.

13. Construct a right isosceles triangle equal in area to a given triangle.

14. Construct a line segment whose square is one-third the product of two given line segments.

15. Construct an equilateral triangle equal in area to a given square.

264. Formulas.

Figure	Formula	
Parallelogram.....	$K = hb$	§ 250
Triangle...........	$K = ab \sin C$	§ 253
	$K = \frac{1}{2}hb$	§ 251
Trapezoid.........	$K = \frac{1}{2}h(b+b')$	§ 260

Review Exercises: Honor Work

1. Find the side of an equilateral triangle the area of which is three times that of an equilateral triangle whose side is s.

2. If $\triangle ABC$ has a right angle at C, prove that the altitude on the hypotenuse equals $\frac{ab}{c}$.

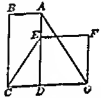
3. The bases of a trapezoid are 20 and 12. If the legs are extended until they meet, the triangle formed on the shorter base as a side has an area of 54. Find the area of the trapezoid.

4. Two adjacent sides of a parallelogram are 10 and 12. If the area is 60, prove that their included angle is 30°.

5. Find the area of a rhombus whose diagonals are 16 and 20.

6. Cut off one-fifth of a parallelogram by drawing a line through one of the vertices.

7. If rectangle $ABCD$ is equal in area to square $EDGF$, prove that $\triangle EDC$ is similar to $\triangle ADG$.



8. If the ends of one side of a parallelogram are joined to any point on the opposite side, prove that a triangle is formed whose area is one-half that of the parallelogram.

9. If a vertex of a parallelogram and the middle points of the opposite sides are joined, the triangle formed is three-eighths of the parallelogram.

10. The bases of a trapezoid are 10 and 24, and the lower base angles are each 45 degrees. Find the area.

11. Find the area of a trapezoid whose bases are 8 and 24, and whose lower base angles are each 60°.

12. Bisect a parallelogram by a line perpendicular to the base.

13. If two triangles are similar, construct a triangle similar to them and equal to their difference.

14. The area of a rhombus is 72, and one diagonal is 16. Find the other diagonal and a side.

15. Find the side of an equilateral triangle whose area equals the sum of the areas of two equilateral triangles whose sides are 6 and 8.

16. Two sides of a triangle are 10 and 12 and their included angle is 150°. Find the area.

17. Similar triangles are to each other as the squares of the bisectors of corresponding angles.

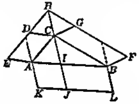
18. The area of an equilateral triangle, whose side is s, is $\frac{s^2}{4}\sqrt{3}$.

19. If P is on the diagonal BD of $\square ABCD$, then the $\triangle PAB$ and PBC are equal in area.

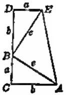
20. One diagonal of a rhombus is 2 greater than the other and their sum is 14. Find the side of the rhombus.

21. One diagonal of a rhombus exceeds the other by 14, and a side of the rhombus is 35. Find the diagonals.

22. $\square ACDE$ and $\square CBFG$ are constructed on the sides of $\triangle ABC$, and the sides ED and FG are extended until they meet at H. Then $\square AKLB$ is constructed on AB with AK equal and parallel to HC. Prove that $AKLB = ACDE + CBFG$.



Ex 22



Ex. 24.

23. Construct a line parallel to a given line so that it will bisect a given parallelogram.

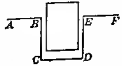
24. President Garfield discovered the following proof that the square of the hypotenuse of a right triangle equals the sum of the squares of the legs. Given right $\triangle ABC$, he extended CB to D making BD equal to AC, and constructed a right angle at D. He then took DE equal to CB, and drew EB and EA. He proved $\triangle EBA$ a right triangle and $EDCA$ a trapezoid. Finally he equated the area of the trapezoid to the sum of the three triangles, and simplified the result. Write his complete proof.

Applied Problems

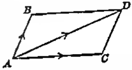
25. To an observer at F, the image E of C, as seen in a plane mirror AB, appears to be at E, on CD the perpendicular to AB, with the distances CD and DE equal. The light from C actually strikes the mirror at H and is reflected to F. If GH is drawn

perpendicular to AB, prove that "the angle of incidence," $\angle CHG$, equals the "angle of reflection," $\angle GHF$.

26. To continue a straight line AB on the other side of a house, a surveyor constructs right angles at B, C, D, and E, and makes the lengths BC, CD and DE each 50 ft. Prove that EF will lie in the same straight line with AB.



27. If two forces acting on point A are represented, in direction and amount, by AB and AC, the diagonal AD of $\square ACDB$ will represent "the resultant" in direction and amount, the resultant being a single force which would have the same effect as the two forces. If a force of 10 lbs. acts at right angles to a force of 24 lbs., find the amount of the resultant.



28. If two forces of 10 lbs. each act at an angle of 60° with each other, prove that their resultant bisects the angle formed by their directions, and find its amount.

BOOK FIVE

POLYGONS

265. In § 38, a polygon was defined as a figure formed by straight lines which enclose a portion of the plane. This includes all of the shapes shown below:

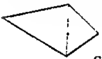


Fig. 1.

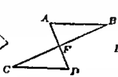


Fig. 2.

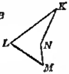


Fig. 3.

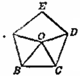


Fig. 4.

In this book, we do not in general deal with polygons such as that in Fig. 2, in which the sides cross between the vertices, nor with polygons such as that in Fig. 3, in which sides can be extended into the space inside the polygon. Most of our propositions, however, are equally true for these cases.

Can you give a proposition from Book One which is not true for quadrilaterals such as that in Fig. 2?

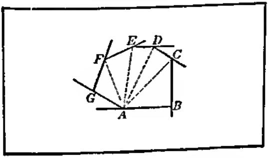
A diagonal of a polygon is a straight line, not a side, which joins two of the vertices.

The following is a table of the important polygons:

Name of Polygon	Number of Sides
Triangle	3
Quadrilateral	4
Pentagon	5
Hexagon	6
Octagon	8
Decagon	10

Proposition 1

266. *The sum of the angles of a polygon of n sides is* $(n-2)$ *straight angles.*



Given: Polygon $ABCDE \ldots$ having n sides.

To prove: The sum of the angles of $ABCDE \ldots$ is $(n-2)$ straight angles.

Proof:

Statements	Reasons
1. Draw all the diagonals from A, forming $(n-2)$ triangles.	1. Post. 1.
2. The sum of the $\angle$s of a $\triangle = 1$ st. $\angle$.	2. § 86.
3. The sum of the $\angle$s of $(n-2)\triangle$s $=(n-2)$ st. $\angle$s.	3. Ax. 5.
4. The sum of the $\angle$s of $(n-2)\triangle$s equals the sum of the $\angle$s of $ABCDE \ldots$	4. Ax. 7.
5. The sum of the $\angle$s of $ABCDE \ldots = (n-2)$ st. $\angle$s.	5. Ax. 1.

267. Corollary 1. *The sum of the exterior angles of a polygon, formed by extending in succession one side at each vertex, is two straight angles.*

The sum of the ext. and int. $\angle$s of $ABCDE \ldots$ is n st. $\angle$s.
The sum of the int. $\angle$s of $ABCDE \ldots$ is $(n-2)$ st. $\angle$s. (§ 266.)
The sum of the ext. $\angle$s of $ABCDE \ldots$ is 2 st. $\angle$s. (Ax. 4.)

268. Corollary 2. *Each angle of an equiangular polygon of n sides equals* $\frac{n-2}{n}$ *straight angles.*

269. *In a series of equal fractions, if the sum of any number of the numerators is divided by the sum of the corresponding denominators, the quotient equals any one of the fractions.*

Given: $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}=\ldots$

To prove: $\frac{a+c+e+\ldots}{b+d+f+\ldots}=\frac{a}{b}=\frac{c}{d}=\ldots.$

Let $\frac{a}{b}=\frac{c}{d}=\ldots=r$. Then $a=br$, $c=dr$, $e=fr$, ... (Ax. 5).

$a+c+e+\ldots=(b+d+f+\ldots)r$ (Ax. 3).

$\frac{a+c+e+\ldots}{b+d+f+\ldots}=r=\frac{a}{b}=\frac{c}{d}=\ldots$ (Ax. 6 and 2).

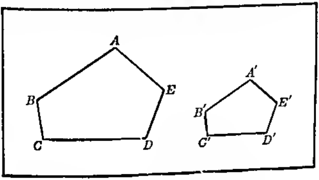
Exercises

1. Define a *polygon; similar polygons.*
2. What is the *perimeter* of a polygon?
3. State two propositions about an exterior angle of a triangle. Are these propositions equally true for quadrilaterals? Other polygons?
4. Can you draw a polygon in which every exterior angle equals every interior angle? One in which every exterior angle is smaller than every interior angle?

Investigation Problem. Two triangles ABC and $A'B'C'$ are similar, and $B'C'$ is twice BC. Compare their perimeters. If BC were n times $B'C'$, how would the perimeters compare? Does your conclusion hold equally for similar quadrilaterals? For other similar polygons? If a, b, c, ... are the sides of one polygon and a', b', c', ... the sides of a similar polygon, write a proportion in these letters. Does the fraction $\frac{a+b+c+\ldots}{a'+b'+c'+\ldots}$ equal one of these ratios? Why?

Proposition 2

270. *The perimeters of two similar polygons are to each other as any pair of corresponding sides.*



Given: $ABCDE \ldots \sim A'B'C'D'E' \ldots$ with perimeters P and P'.

To prove: $\dfrac{P}{P'}=\dfrac{AB}{A'B'}=\dfrac{BC}{B'C'}=\ldots$

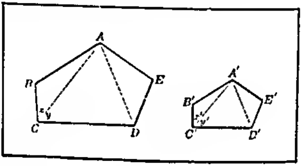
Proof:

Statements	Reasons
1. $\dfrac{AB}{A'B'}=\dfrac{BC}{B'C'}=\dfrac{CD}{C'D'}=\ldots$	1. § 215.
2. $\dfrac{AB+BC+CD+\ldots}{A'B'+B'C'+C'D'+\ldots}=\dfrac{AB}{A'B'}=\dfrac{BC}{B'C'}=\ldots$	2. § 269.
3. $P=AB+BC+CD+\ldots$ and $P'=A'B'+B'C'+\ldots$	3. § 38.
4. $\dfrac{P}{P'}=\dfrac{AB}{A'B'}=\dfrac{BC}{B'C'}=\ldots$	4. Ax. 1.

Investigation Problem. If AC and $A'C'$ were drawn in the above similar polygons, would $\triangle ABC$ be similar to $\triangle A'B'C'$? What parts of these triangles are parts of the polygons? Are these sufficient to prove the triangles similar? Now draw AD and $A'D'$. Are $\triangle ACD$ and $A'C'D'$ similar? Can you prove $\angle ACD$ equal to $\angle A'C'D'$? Can you find a ratio to which both $\dfrac{AC}{A'C'}$ and $\dfrac{CD}{C'D'}$ are equal? Is the converse true?

Proposition 3

★ **271.** *Similar polygons can be cut into triangles so that the corresponding triangles are similar.*



Given: $ABCDE \ldots \sim A'B'C'D'E' \ldots$

To prove: $ABCDE \ldots$ and $A'B'C'D'E' \ldots$ can be cut into triangles so that the corresponding triangles are similar.

Proof:

Statements	Reasons
1. Draw all the diagonals from A and A'.	1. Post. 1.
2. $\angle B = \angle B'$.	2. § 215.
3. $AB : A'B' = BC : B'C'$.	3. § 215.
4. $\triangle ABC \sim \triangle A'B'C'$.	4. § 220.
5. $\angle BCD = \angle B'C'D'$ and $\angle x = \angle x'$.	5. § 215.
6. $\angle y = \angle y'$.	6. Ax. 4.
7. $AC : A'C' = BC : B'C'$ and $CD : C'D' = BC : B'C'$.	7. § 215.
8. $AC : A'C' = CD : C'D'$.	8. Ax. 2.
9. $\triangle ACD \sim \triangle A'C'D'$.	9. § 220.

In the same way the other pairs of triangles can be proved similar.

272. Converse. *Polygons are similar, if they can be cut into triangles so that the corresponding triangles are similar, and similarly placed.*

Proof:

STATEMENTS	REASONS
1. $\angle B = \angle B'$, $\angle x = \angle x'$, $\angle y = \angle y'$, ...	1. § 215.
2. $\angle C = \angle C'$, $\angle D = \angle D'$, ...	2. Ax. 3.
3. $AB : A'B' = BC : B'C' = (AC : A'C') = CD : C'D' = \ldots$	3. § 215.
4. $ABCDE \ldots \sim A'B'C'D'E' \ldots$	4. § 215.

EXERCISES

1. If two polygons are similar, and lines are drawn joining in order the middle points of the sides of each, the corresponding triangles cut off are similar.

2. Using the same hypothesis, prove that the new polygons formed inside are similar.

3. If two similar polygons are cut into triangles by diagonals so that one triangle of one of them is congruent to the corresponding triangle of the other, then all corresponding triangles are congruent and the polygons are congruent.

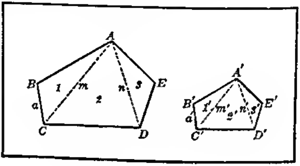
4. Polygons $ABCDE$ and $A'B'C'D'E'$ are similar, and P and P' are points inside them respectively so located that $\triangle PAB$ is similar to $\triangle P'A'B'$. Prove that $\triangle PBC$ is similar to $\triangle P'B'C'$. Also that lines from these points to all the vertices of their respective polygons cut the polygons into similar triangles.

5. If two pentagons have their corresponding angles equal, what is the smallest number of sides of one which must be known to be proportional to corresponding sides of the other in order that the polygons can be proved similar? Prove your conclusion.

Investigation Problem. In the similar polygons above, compare the ratio of the areas of $\triangle ABC$ and $\triangle A'B'C'$ to the ratio of the diagonals AC and $A'C'$. Then compare the ratio of the areas of $\triangle ACD$ and $A'C'D'$ to the ratio of these same diagonals. The sum of the triangles of one polygon is to the sum of the triangles of the other polygons as . . . Why? What conclusion would you draw about the areas of the polygons? Now complete and prove the statement, "The areas of similar polygons are to each other as . . . "

Proposition 4

★ 273. *The areas of similar polygons are to each other as the squares of corresponding sides.*



Given: $ABCDE \ldots \sim A'B'C'D'E' \ldots$

To prove: $\dfrac{ABCDE \;\ldots}{A'B'C'D'E' \ldots} = \dfrac{a^2}{a'^2}$.

Proof:

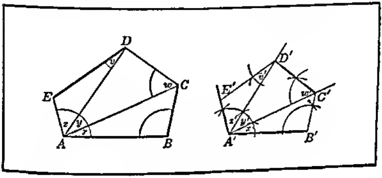
Statements	Reasons
1. Construct all the diagonals from A and A'.	1. Post. 1.
2. $\Delta 1 \sim \Delta 1'$, $\Delta 2 \sim \Delta 2'$, $\Delta 3 \sim \Delta 3'$, ...	2. § 271.
3. $\dfrac{a^2}{a'^2} = \dfrac{\Delta 1}{\Delta 1'} = \dfrac{m^2}{m'^2} = \dfrac{\Delta 2}{\Delta 2'} = \dfrac{n^2}{n'^2} = \dfrac{\Delta 3}{\Delta 3'} = \ldots$	3. § 261.
4. $\dfrac{\Delta 1 + \Delta 2 + \Delta 3 + \ldots}{\Delta 1' + \Delta 2' + \Delta 3' + \ldots} = \dfrac{a^2}{a'^2}$.	4. § 269.
5. $\dfrac{ABCDE \;\ldots}{A'B'C'D'E' \ldots} = \dfrac{a^2}{a'^2}$.	5. Ax. 1.

274. In general, the areas of similar figures are to each other as the squares of any pair of corresponding lines.

This proposition has many important applications in physics. Because of it, in gravitation, light, sound, electric waves, and magnetism, the intensity at any particular point varies inversely as the square of its distance from the source.

Proposition 5

275. *A polygon similar to a given polygon can be constructed.*

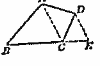


Given: Polygon $ABCDE$ and line $A'B'$.

To prove: A polygon can be constructed on $A'B' \sim$ to $ABCDE$.

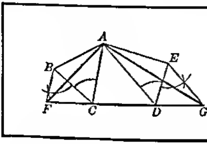
Construction: Statements	Reasons
1. Draw AC and AD.	1. Post. 1
2. At A' construct $\angle s\ x', y'$ and $z' = \angle s\ x, y,$ and z.	2. § 66.
3. Construct $\angle B' = \angle B$, then $\angle w' = w$, then $\angle v' = \angle v$. $A'B'C'D'E'$ is the required polygon.	3. § 66.
Proof:	
1. $\triangle A'B'C' \sim \triangle ABC$, $\triangle A'C'D' \sim \triangle ACD$, $\triangle A'D'E' \sim \triangle ADE$.	1. § 216.
2. $A'B'C'D'E' \sim ABCDE$.	2. § 272.

Investigation Problem. Can you construct a triangle having the same area as quadrilateral $ABCD$? Try to replace $\triangle ACD$ with another triangle having the same area. If we keep AC as the base, what is the locus of point D? To what point must D move so that CD forms a straight line with BC? What triangle does $\triangle ACD$ then become? What does quadrilateral $ABCD$ then become?



Proposition 6

276. *A triangle equal in area to a given polygon can be constructed.*



Given: Polygon $ABCDE$.

To prove: A triangle equal to $ABCDE$ can be constructed.

Construction: Statements	Reasons
1. Extend CD.	1. Post. 2.
2. Draw AD.	2. Post. 1.
3. Construct $EG \parallel AD$.	3. § 77.
4. Draw AG.	4. Post. 1
Proof:	
1. In $\triangle ADE$ and ADG, AD is the base of both.	1. Iden.
2. E and G are on a line $EG \parallel AD$.	2. Const.
3. $\triangle ADE$ and $\triangle ADG$ are equal in area.	3. § 262.
4. $ABCDE = ABCG$, and the number of sides is reduced by one.	4. Ax. 1.

In the same way, the number of sides can be reduced one at a time, until $\triangle AFG$ is obtained. Therefore $\triangle AFG = ABCDE$ (Ax. 1).

Exercises

1. Find the number of straight angles in the sum of the angles of a quadrilateral; of a hexagon; of an octagon; of a decagon.

2. Find the number of degrees in the sum of the angles of a polygon of 7 sides; of 12 sides.

3. Find one angle of an equiangular polygon of 4 sides; of 6 sides; of 10 sides; of 60 sides.

4. Find the number of degrees in an exterior angle of an equiangular polygon of 5 sides; of 7 sides; of n sides.

5. Find the number of sides of a polygon, the sum of whose angles is 2 straight angles; 6 straight angles; 20 right angles; 100 right angles; 540°; 1080°.

6. Find the number of sides of a polygon, each of whose angles is 150°; 175°; 162°; 135°; 179°.

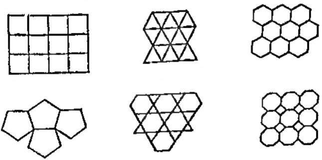
7. Find the number of sides of a polygon, each of whose exterior angles is 40°; 15°; 90°.

8. Find the number of sides of a polygon, each of whose interior angles is 11 times the adjacent exterior angle.

9. If two angles of a quadrilateral are supplementary, the other two are supplementary.

10. If the sum of four of the angles of a pentagon is 500°, find the fifth angle.

11. If, from any point D inside the $\angle ABC$, perpendiculars are drawn to the sides AB and BC, their included $\angle D$ will be the supplement of $\angle ABC$.



12. Tiled floors are often made of regular (equilateral and equiangular) polygons. Show that triangles, squares, or hexagons will fit together to form such tiling, but that pentagons or octagons will not fit.

13. Show that two such octagons and a square can be used to fill the angular space around a point. (In order that the figures fit around a point, the sum of the angles at that point must equal 360°.)

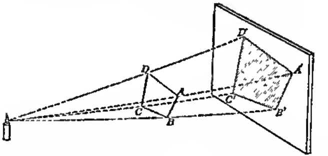
14. Prove that similar polygons are to each other as the squares of their corresponding diagonals.

15. The sides of a polygon are 8, 11, 7, 4, and 13, its area is 120, and one diagonal is 10. If the side of a similar polygon corresponding to 8 is 12, find the other sides, the corresponding diagonal, and the area.

16. The areas of two similar polygons are 64 and 25. Compare the lengths of their sides.

Optional Exercises

17. The area of a polygon is 20 and one side is 8. The area of a similar polygon is 12. Find the corresponding side.



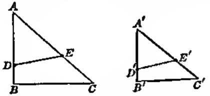
18. If $ABCD$ is a polygon and $A'B'C'D'$ its shadow on a wall parallel to it, show that $ABCD$ and $A'B'C'D'$ are proportional to the squares of their distances from the light. (Assume that $ABCD$ and $A'B'C'D'$ are similar and that $OB' \perp AB$ and $A'B'$.)

19. If, in Ex. 18, $ABCD$ is one-third of the way from the candle to the wall, and has an area of 8 sq. in., what is the area of its shadow?

20. One side of a polygon is 10. Find the corresponding side of a similar polygon whose area is twice as great; n times as great.

21. Three polygons are similar, and corresponding sides of two of them are 6 and 8. If the third polygon equals their sum, find its corresponding side.

22. Construct a polygon similar to two given similar polygons and equal in area to their sum; to their difference.

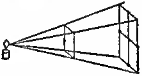


Ex. 23.

23. If $\triangle ABC$ is similar to $\triangle A'B'C'$, and $\frac{AD}{A'D'} = \frac{AE}{A'E'} = \frac{AB}{A'B'}$, prove that $BCED$ is similar to $B'C'E'D'$.

Applied Problems

24. Since light from a point travels in straight lines, the same amount of light that would strike a polygon 1 ft. away will strike at a distance of 2 ft. a similar polygon whose corresponding sides are twice as long. How will their areas compare?



25. If the same amount of light is spread out over twice as great an area, we say that the intensity is half as great. How would the intensities compare in Ex. 24?

26. When you study your lesson in the evening, if you hold your book only 3 ft. from the light, how many times as great is the intensity of the illumination as it would be if you held the book 12 ft. from the light?

27. A woman finds that she has used 2 yds. of cloth in making a dress for her little daughter. How much cloth must she use to make a similar dress for her other daughter who is one-and-a-half times as tall?

28. Mrs. Hartney wants to estimate the number of yards of

cloth she must buy to make a dress. She has a small picture showing the separate pieces of the pattern, and she finds that she can lay these on a rectangle of paper containing 6 sq. in. She also finds that a piece, which in the real pattern would be 2 ft. long, has a length of only 1.2 in. in her picture of it. Estimate the number of square yards she will need for the dress.

29. On a map of the United States, 1 ft. represents 1,000 miles. If the area of the map is 3 sq. ft., what is the area of the United States?

30. If the State of New York has an area of 45,000 sq. mi., what area will it occupy on this map?

31. A geography publisher wishes to illustrate by areas of similar figures the comparative amounts of wheat produced in different countries. If the ratio for two countries is 4 to 1, what ratio of the sides of his figures should he use?

32. One country produces three times as much wheat as a second. The first he represents by a polygon whose base is 1 in. What length should he take as the base of the similar polygon which represents the second?

33. In an advertisement, a newspaper compares its circulation with that of its competitors by similar diagrams. The height of the figure for its own circulation is twice that for a rival's. Does this properly show the relationship, if its circulation is twice that of its rival? Explain.

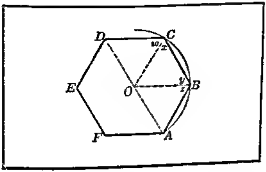
Regular Polygons

277. A **regular polygon** is a polygon which is both equilateral and equiangular.

Investigation Problem. Do you think a circle could be circumscribed about any one of these regular figures? Let us try to prove it in the case of the regular polygon. Can you pass a $\odot O$ through the three vertices A, B, and C? What must you now know about OD in order to prove that your circle will pass through point D? Can you prove OD equal to OA?

Proposition 7

★278. *A circle can be circumscribed about any regular polygon.*



Given: Regular polygon $ABCDE\ldots$.

To prove: That a circle can be circumscribed about $ABCDE\ldots$.

Proof:

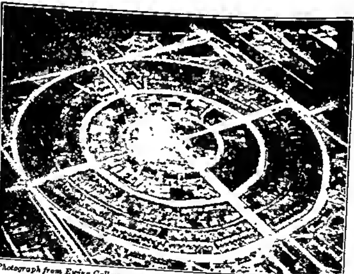
Statements	Reasons
1. Construct a $\odot$ through A, B, and C, and let O be its center.	1. § 175.
2. Draw OA, OB, OC, and OD.	2. Post. 1.
3. In $\triangle OAB$ and ODC, $AB = DC$.	3. § 277.
4. $OB = OC$.	4. § 138.
5. $\angle y = \angle x$.	5. § 55.
6. $\angle ABC = \angle DCB$.	6. § 277.
7. $\angle z = \angle w$.	7. Ax. 4.
8. $\triangle OAB \cong \triangle ODC$.	8. § 50.
9. $OA = OD$, or $OD =$ the radius.	9. § 22.
10. $\odot O$ passes through D.	10. § 140.
In like manner, the $\odot O$ can be shown to pass through the other vertices.	
11. $\odot O$ is circumscribed about $ABCDE\ldots$	11. § 136.

279. Corollary 1. *A circle can be inscribed in any regular polygon.*

Circumscribe a ○ (§ 278). Then equal chords are equally distant from the center (§ 149). Then use § 153.

280. The center of a regular polygon is the center of its circumscribed and inscribed circles. The radius of the circumscribed circle is the radius of the polygon. The radius of the inscribed circle is called the apothem of the polygon.

281. Corollary 2. *The central angle forward by two consecutive radii of a regular polygon of n sides equals* $\left(\frac{360}{n}\right)$ *degrees.*



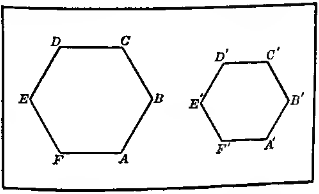
Photograph from Ewing Galloway.

GEOMETRY PLAYS A PART IN MODERN TOWN PLANNING

This shows the principle of a circle inscribed in a square as it was worked out in a model town in England. Can you see arcs, segments, diameters, and tangents in this picture?

Proposition 8

282. *Regular polygons of the same number of sides are similar.*



Given: Regular polygons $ABCDE\ldots$ and $A'B'C'D'E'\ldots$ having the same number of sides.

To prove: $ABCDE\ldots \sim A'B'C'D'E'\ldots$

Proof:

Statements	Reasons
1. $\angle A = \angle B = \angle C = \ldots = \frac{n-2}{n}$ st. $\angle$s.	1. § 268.
2. $\angle A' = \angle B' = \angle C' = \ldots = \frac{n-2}{n}$ st. $\angle$s.	2. § 268.
3. $\angle A = \angle A'$, $\angle B = \angle B'$, ...	3. Ax. 2.
4. $AB = BC = CD = \ldots.$	4. § 277.
5. $A'B' = B'C' = C'D' = \ldots.$	5. § 277.
6. $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \ldots.$	6. Ax. 6.
7. $ABCDE\ldots. \sim A'B'C'D'E'\ldots$	7. § 215.

Ex. If two regular polygons have the same number of sides, do the lines from their centers to their vertices cut them into similar triangles? Prove your conclusion.

283. To avoid the usual difficult proofs for the circumference and area of a circle, we shall assume as an axiom the following:

284. *A theorem, true for regular polygons, and independent of the number of sides, is also true for circles.*

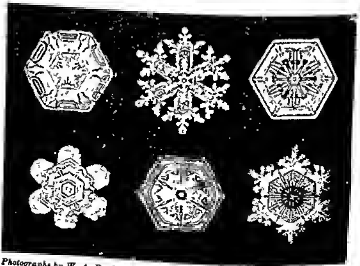
Exercises

1. Find the number of degrees in each angle of a regular polygon of 6 sides; 8 sides; 12 sides.

2. Find the sum of the angles of each of the above polygons.

3. The side of a regular polygon is 8 and its perimeter is 112. Find the side of a regular polygon of the same number of sides, if its perimeter is 336.

4. If the diagonals AE and BF of regular hexagon $ABCDEF$ intersect in G, find the number of degrees in $\angle AGF$.

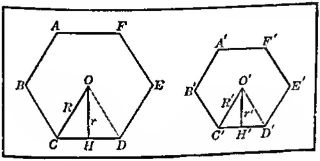


Photographs by W. A. Bentley.

NEARLY ALL SNOWFLAKES ARE HEXAGONAL IN FORM.

PROPOSITION 9

285. *The perimeters of regular polygons of the same number of sides are to each other as their radii, and as their apothems.*



Given: Regular polygons O and O', having the same number of sides, with radii R and R', perimeters p and p', and apothems r and r'.

To prove: $\frac{p}{p'}=\frac{R}{R'}=\frac{r}{r'}$

Proof:

STATEMENTS	REASONS
1. Draw OD and $O'D'$.	1. Post. 1
2. In $\triangle COD$ and $C'O'D'$, $\angle COD=\frac{360°}{n}=\angle C'O'D'$.	2. § 281.
3. $CO=OD$ and $C'O'=O'D'$.	3. § 138.
4. $\frac{CO}{C'O'}=\frac{OD}{O'D'}$.	4. Ax. 6.
5. $\triangle COD \sim \triangle C'O'D'$.	5. § 220.
6. $\frac{R}{R'}=\frac{r}{r'}=\frac{CD}{C'D'}$.	6. § 219.
7. Polygon $O \sim$ polygon O'.	7. § 282.
8. $\frac{p}{p'}=\frac{CD}{C'D'}$.	8. § 270.
9. $\frac{p}{p'}=\frac{R}{R'}=\frac{r}{r'}$.	9. Ax. 2.

286. Corollary 1. *Circumferences are to each other as their radii.* (§§ 285 and 284.)

$$\frac{C}{C'}=\frac{r}{r'}.$$

287. Corollary 2. *The ratio of the circumference of a circle to its diameter is constant.*

$$\frac{C}{C'}=\frac{r}{r'}=\frac{2r}{2r'}=\frac{D}{D'},\quad CD'=C'D,\quad \frac{C}{D}=\frac{C'}{D'}.$$

288. ***The ratio of the circumference to the diameter of a circle is represented by the Greek letter π.*** So,

$$C=\pi D$$

and

$$C=2\pi r$$

The value of π has been found to over 700 decimal places [1] To four places, it is

$$\pi=3.1416$$

or approximately

$$\pi=3\tfrac{1}{7}$$

Why is it necessary to know that the ratio of the circumference to the diameter is constant before defining π? Why do we not use a letter to represent the ratio of the perimeter of any polygon to one of its diagonals?

289. **A sector of a circle is the figure formed by two radii and their intercepted arc.**

Investigation Problem. In the figure of Proposition 9, what is the area of $\triangle OCD$? Draw OB. What is the area of $\triangle OCB$? Compare the altitudes of these triangles. Are the same statements true for the $\triangle OBA$, OAF, etc.? Can you now state and prove a proposition about the area of a regular polygon? What would be the corresponding proposition about the area of a circle?

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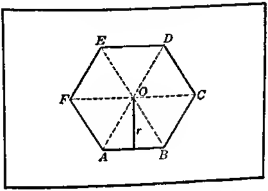
[1] See the supplement, §§ 326 and 327, for the method of calculating the value of π.

Proposition 10

*290. *The area of a regular polygon equals half the product of its perimeter and apothem.*



Given: Regular polygon $ABCDE$ . . . perimeter p and apothem r.

To prove: $ABCDE \ldots = \frac{1}{2}rp$.

Proof:

Statements	Reasons
1. Draw the radii OA, OB, OC, . . .	1. Post. 1.
2. All the triangles formed have the same altitude, r.	2. § 149.
3. $\triangle OAB = \frac{1}{2}r \cdot AB$. $\triangle OBC = \frac{1}{2}r \cdot BC$. $\triangle OCD = \frac{1}{2}r \cdot CD$.	3. § 254.
4. $\triangle OAB + \triangle OBC + \triangle OCD + \ldots = \frac{1}{2}r(AB + BC + CD + \ldots)$.	4. Ax. 3.
5. $ABCDE \ldots = \frac{1}{2}rp$.	5. Ax. 1.

291. Corollary 1. *The area of a circle equals half the product of its radius and circumference.*

$$K = \frac{1}{2}rC \quad (\S\, 284)$$

292. Corollary 2. *The area of a circle equals π times the square of the radius.*

$$K = \tfrac{1}{2}r \cdot 2\pi r \quad (\S 288)$$
$$K = \pi r^2$$

293. Corollary 3. *Circles are to each other as the squares of their radii.*

Class Exercises

In these exercises, choose the value of π so that the result will be correct to as many significant figures as the data used.

1. The area of a circle equals $\frac{1}{4}\pi D^2$, where D is the diameter.
2. Find the area of a circle whose radius is 12; 8.42; 15.693.
3. Find the circumference of a circle whose diameter is 21; 6.2832.
4. Find the area of a circle whose circumference is 22; 8.64; 12.5664.
5. Find the radius of a circle whose area is $314\frac{2}{7}$.
6. Find the circumference of a circle whose area is 15,394.
7. Find the radius of a circle whose circumference is 50; 25.133.
8. Find the radius of a circle whose area is 144π.
9. Find the area of a semicircle whose diameter is 8.

In the following exercises, it is not necessary to extract roots or substitute a value for π.

10. If the radius of one circle is 5 times that of another, find the ratio of their areas; of their circumferences.
11. Find the circumference of a circle having twice the area of a circle whose circumference is 5.
12. Find the radius of a circle equal in area to a square whose side is 8.
13. The area of a circular ring is 628.32 sq. in., and the internal diameter of the ring is 10 in. Find the width of the ring.
14. A square is inscribed in a circle whose radius is 7. Find the area of the segment cut off by a side of the square.

15. If the area of a square is 25, find the area of the circumscribed circle.

16. If the base of a rectangle is 12 and its area 60, find the area of its circumscribed circle.

17. The sides of a triangle are 6, 8, and 10. Find the radius of a circle equal in area to the triangle.

18. A circle and a square each have a perimeter of 100. Compare their areas.

19. If the radius of a circle is 10, find the length of an arc of $90°$; $45°$; $30°$; $20°$; $1°$.

20. If the radius of a circle is 10, find the number of degrees in an arc whose length is 10π; 5π; 10.

21. Find the radius of a circle, if an arc of $40°$ has a length of 10.

22. A square is inscribed in a circle whose radius is 10. Find the part of the circle that is outside the square.

23. Find the area and perimeter of a semicircle whose arc is 12π.

Optional Exercises

24. Find the area of a semicircle whose perimeter is $25\frac{5}{7}$.

25. Find the area of the circle circumscribed about a regular hexagon, the radius of whose inscribed circle is 5.

26. Find the radius of a circle, if the number of square units in its area is 5 times the number of linear units in its circumference.

27. If the altitude of an equilateral triangle is 12, find the areas of its circumscribed and inscribed circles.

28. If the radius of a circle is doubled, state the change in the circumference; in the area.

29. Find the side of an equilateral triangle inscribed in a circle whose area is 64π.

30. How much does the area of an equilateral triangle whose side is 8 exceed the area of its inscribed circle?

31. The sides of a triangle are 9, 10, and 9. Find the radius of a circle equal in area.

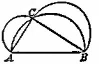
Honor Work

32. If the areas of two circles are to each other as 25 : 16, and the radius of the larger is 10, find the radius of the smaller.

33. Find the area contained between three equal circles, each of which is tangent externally to the other two, and whose radii are each 10 in.

34. The area of a circular ring equals the area of a circle whose diameter is the chord of the outer circle tangent to the inner circle.

35. If semicircles are drawn on the three sides of a right triangle as diameters, as shown in the figure, prove that the sum of the two crescents equals the triangle.



36. If the radius of a circle increases, what change takes place in the circumference? in the area? When the radius has increased 4 in., how much has the circumference increased?

37. Find the area of that part of a circle, radius 8, that is outside its inscribed equilateral triangle.

Applied Problems

38. How many revolutions does a bicycle wheel, whose diameter is 28 in., make in going a mile?

39. A circular drive is 15 ft. wide, and the enclosed grass plot is 80 ft. in diameter. Find the area of the drive.

40. The minute hand and the hour hand of a watch are $\frac{3}{4}$ in. long and $\frac{1}{2}$ in. long respectively. How much farther does the end of the minute hand travel in a day than the end of the hour hand?

41. A circular pond 100 yds. in diameter is surrounded by a walk 10 ft. wide. Find the cost of paving the walk at 60 cts. a square yard.

42. If the amount of water flowing through a pipe is proportional to the area of cross section, how many times as much water will flow through a 3-in. pipe as through a 1-in. pipe?

43. If an aqueduct 10 in. in diameter supplies a city of 12,000

people with water, what population will one 15 in. in diameter supply?

44. Two branches of a sewer pipe are respectively $3\frac{1}{2}$ and $4\frac{1}{2}$ in. in diameter. What must be the diameter, to the nearest inch, of the pipe into which they empty, in order that the sewage may be carried off?

45. If, from a sheet of tin 3 ft., 4 in. by 1 ft., 4 in., the greatest number of circular disks 8 in. in diameter are cut, how much of the tin is wasted?

46. The diameter of a driving pulley is 15 in. and its speed is 375 revolutions per minute. What is the speed of the driven pulley whose diameter is 5 in.?

47. A driving pulley 6 in. in diameter has a speed of 90 revolutions per minute. If the driven pulley is to turn 75 revolutions per minute, what diameter must it have?

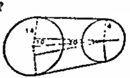
48. The piston of a steam engine is 20 in. in diameter. If the steam gauge registers a pressure of 90 lbs. per sq. in., what is the total pressure on the piston?

49. The pedal sprocket wheel of a bicycle has 26 cogs, and the rear sprocket wheel has 8 cogs. If the diameter of the wheel is 28 in., how far will the bicycle travel during one complete turn of the pedals?

50. The gear of a bicycle is the product of the diameter of the wheel by the number of times the wheel turns around to each turn of the pedals. How far will a bicycle with a gear of 90 travel while the pedals turn around once?

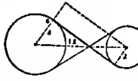
51. Assuming that the earth travels around the sun in a circle of 93,000,000 miles radius in 365 days, how many miles per hour does it average? How many miles per second?

52. Find the length of the belt passing around two pulleys 20 in. apart, and having radii of 14 in. and 4 in.



53. Katherine Long wishes to buy some binding for the edge of her hat which is 1 ft. in diameter. What length will she need?

54. Two pulleys 12 ft. apart are connected by a belt, crossed as shown in the figure. If the radii of the pulleys are 4 ft. and 2 ft. respectively, find the total length of the belt.



55. The electrical resistance of a wire varies inversely as the area of a cross section, that is, a wire with a cross section of 2 sq. in. would have only half the resistance of one whose cross section was 1 sq. in. If a copper wire 100 meters long and 1 millimeter in diameter has a resistance of 2 ohms, find the resistance of the same length of copper wire whose diameter is 2 millimeters; 5 millimeters; $\frac{1}{2}$ millimeter.

56. A china decorator wishes to paint around the edge of a plate a band containing 30 small figures equally spaced. Find to the nearest hundredth of an inch the distance between their centers, if the diameter of the plate is 9 in.

57. If a like band of figures, having the same spacing, is painted around the saucer whose diameter is 5.4 in., how many figures will be required?

58. Mrs. Hamilton crochets circular doilies 1 ft. in diameter which she sells for $1 each. A customer wants to know what price she would charge to make a doily 20 in. in diameter. If the cost is proportional to the area, what price should she ask?

59. A friend of Mrs. Leonard made a circular renaissance lace center piece for her dining table, which Mrs. Leonard admired and decided to copy. The lace, constructed around a circular center of linen 30 in. in diameter, is 10 in. wide, and required 24 yds. of braid. How much braid will Mrs. Leonard need, if she makes the lace 12 in. wide around a circular center of linen 36 in. in diameter?

60. A newspaper wishes to compare the cost of public school education in the various states, by the areas of circles. If Tennessee, which spends $15,000,000, is represented by a circle whose diameter is 1 in., what radius should be used for the circle representing Nebraska which spends $30,000,000? If California spends $90,000,000, what length radius should be taken for its circle?

Sectors and Segments

61. What part of a circle is a sector whose angle is 90°? 60°? 40°? 80°? 63°? n°?

62. In a circle whose radius is 10, find the area of the sector whose angle is 60°; 45°; 100°; 52°; n°.

63. Make a formula for finding the area of a sector whose angle is $n°$, if the radius of the circle is r.

64. Find the angle of a sector if its area is 6π and its radius is 4.

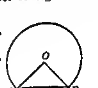
65. The area of a sector, whose angle is 40°, is 4π. What is the radius of the circle?

66. In a circle whose radius is 12, find the length of an arc of 90°; of 120°; of 80°; of 46°; of $x°$.

67. Make a formula for finding the length of an arc of $n°$, if the radius of the circle is r.

68. An arc of 20° is 3 in. long. What is the radius of the circle?

69. In a circle whose radius is 10, find the number of degrees in an arc whose length is 5π; 4π; π; 10; 3.



70. In $\odot O$, $\angle O = 90°$ and $OA = 6$. Find the area of, (a) sector OAB; (b) $\triangle OAB$; (c) segment AB.

71. If $OA = 10$ and $\angle O = 60°$, find the area of segment AB.

72. If the radius of the circle is 8, find the area of a segment whose arc is 120°; 60°; 45°.

73. Is the segment whose arc is 90° equal to twice the segment whose arc is 45°?

74. A cow is tied to a fence at the roadside by a rope $10\sqrt{2}$ ft. long. The road is 10 ft. wide, and there is grass on the other side but none on the side on which the fence is. Over how many square feet of grassland can she graze?

Completion Test (10 min.)

1. The radius of a circle is 10. The radius of a circle, whose area is 4 times that of the given circle is . . . .

2. If the radius of a circle is 6, the length of an arc of 30° is . . . .

3. If the area of a circle is 25π, its radius is . . . .

4. The diameter of the front wheels of a go-cart is half the diameter of the back wheels. In going a certain distance, the back wheels will turn . . . as many times as the front wheels.

5. Doubling the radius of a circle multiplies the area by . . . .

6. The radius of the circle inscribed in an equilateral triangle is . . . of the altitude of the triangle.

7. A bicycle wheel is 28 in. in diameter. In traveling 880 in., it will make . . . revolutions. (Use $\pi = \frac{22}{7}$.)

8. The ratio of the circumference of a circle to the . . . is denoted by the symbol π.

Numerical Test (*10 min.*)

1. The radius of a circle is 6. If the area of a sector is 9π, find the number of degrees in the angle of the sector.

2. The circumference of a circle is 18π. Find its area.

3. In a circle whose radius is 10, find the area of a sector whose arc is 18.

4. If the radius of a circle is 2, find the apothem of a regular inscribed hexagon.

5. If a sector of a circle, whose angle is 30°, has an area of 3π, find the radius.

True-False Test (*10 min.*)

Write T if the statement is true, F if it is false.

1. Each angle of a polygon of n sides equals $\frac{n-2}{n}$ straight angles.

2. The area of a circle equals half the product of its radius and circumference.

3. The areas of similar polygons are to each other as their radii.

4. The area of a triangle equals half its base times its altitude.

5. The areas of similar triangles are to each other as the squares of corresponding sides.

6. If a circle can be inscribed in a polygon, the polygon is regular.

7. The sum of the angles of a regular polygon having n sides is $(n-2)$ st. $\angle$.

8. An inscribed equiangular polygon is regular.

9. The perimeters of similar polygons are to each other as the squares of corresponding sides.

10. The area of a regular polygon equals half its perimeter times its apothem.

MATCHING TEST (*10 min.*)

Copy the numbers. After each write the letter of the phrase which corresponds to the formula following the number.

1. $c=\sqrt{a^2+b^2}$	*a.* Area of a circle.
2. $K=\frac{1}{4}s^2\sqrt{3}$	*b.* Area of a triangle.
3. $K=\frac{1}{2}hb$	*c.* Area of a trapezoid.
4. $h=\frac{1}{2}s\sqrt{3}$	*d.* Area of a sector.
5. $K=\frac{\text{central}\angle}{360}\pi r^2$	*e.* Circumference of circle.
6. $K=\frac{1}{2}h(b+b')$	*f.* Altitude of equilateral triangle.
7. $K=\pi r^2$	*g.* Area of equilateral triangle.
8. $K=\frac{3}{2}s^2\sqrt{3}$	*h.* Area of regular hexagon.
9. $C=2\pi r$	*i.* Diagonal of a square.
10. $d=s\sqrt{2}$	*j.* Hypotenuse of right triangle.

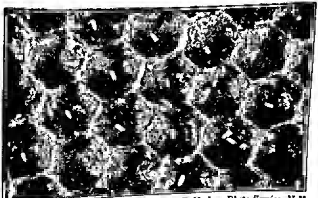
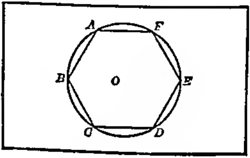


Photo from Publishers Photo Service, N.Y.

THE HONEYCOMB OF BEES IS FORMED OF ALMOST PERFECT HEXAGONS
Notice the eggs in the cells.

Proposition 11

294. *If a circle is divided into any number of equal arcs, the chords of these arcs form a regular inscribed polygon.*



Given: $\odot O$, divided into equal arcs $AB, BC, CD, \ldots$ chords $AB,\ BC,\ CD, \ldots$.

To prove: $ABCD \ldots$ a regular polygon.

Proof:

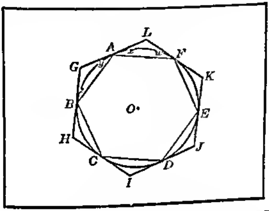
Statements	Reasons
1. $\widehat{AB} = \widehat{BC} = \widehat{CD} = \ldots$.	1. Hyp.
2. Chord AB = chord BC = chord $CD = \ldots$.	2. § 145.
3. $\widehat{FAB} = \widehat{ABC} = \widehat{BCD} = \ldots$.	3. Ax. 3.
4. $\angle A = \angle B = \angle C = \ldots$.	4. § 167.
5. $ABCD \ldots$ is a regular polygon.	5. § 277.

Ex. An equilateral polygon inscribed in a circle is regular.

Investigation Problem. Draw tangents to $\odot O$ at $A, B, C, \ldots$ Does the circumscribed polygon appear to be regular? How can you prove the angles equal? If you knew that the parts of the sides were all equal, could you complete the proof? Are the triangles isosceles? Prove your answer. Are the triangles congruent? Prove your answer. If you knew how to inscribe any certain regular polygon in a circle, could you circumscribe a regular polygon having the same number of sides?

Proposition 12

295. *If a circle is divided into any number of equal arcs, the tangents at the points of division form a regular circumscribed polygon.*



Given: $\odot O$, divided into equal arcs AB, BC, CD, . . .
and tangents GH, HI, IJ, . . . at B, C, D, . .

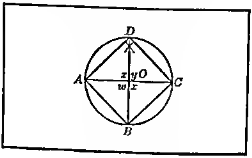
To prove: $GHIJ$ . . . a regular polygon.

Proof: STATEMENTS	REASONS
1. Draw AB, BC, CD, . . .	1. Post.1
2. $\angle w$ is measured by $\frac{1}{2}\widehat{FA}$, $\angle x$ is measured by $\frac{1}{2}\widehat{FA}$, $\angle y$ is measured by $\frac{1}{2}\widehat{AB}$, ...	2. § 170.
3. $\widehat{FA}=\widehat{AB}=\widehat{BC}=$. . . .	3. Hyp.
4. $\angle w=\angle x=\angle y=\angle z=$. . . .	4. Ax. 6.
5. Chord $FA=$ chord $AB=$ chord $BC=$. . . .	5. § 145.
6. $\triangle LFA\cong\triangle GAB\cong\triangle HBC\cong$. . . .	6. § 52.
7. $AL=AG$, $GB=BH$, ...	7. § 22.
8. $FL=AL$, $AG=GB$, ...	8. § 157.
9. $AL+AG=GB+BH=$. . . . or $LG=GH=$. . . .	9. Ax. 3.
10. $\angle L=\angle G=\angle H=$. . . .	10. § 22.
11. $GHIJ$ . . . . is a regular polygon	11. § 277.

Ex. Tangents, parallel to the sides of a regular inscribed polygon form a regular circumscribed polygon.

Proposition 13

296. *A square can be inscribed in a circle.*



Given: $\odot O$.

To prove: A square can be inscribed in $\odot O$.

Construction:

Statements	Reasons
1. Construct diameter AC from point A.	1. Post. 1.
2. Construct diameter $BD \perp AC$.	2. § 64.
3. Draw AB, BC, CD, and DA. $ABCD$ is the required square.	3. Post. 1.

Proof:

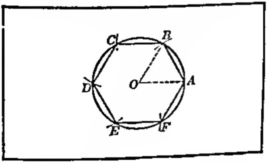
Statements	Reasons
1. $\angle s\ w, x, y$, and z are rt. $\angle s$.	1. § 12.
2. $\angle w = \angle x = \angle y = \angle z$.	2. § 33.
3. $\widehat{AB} = \widehat{BC} = \widehat{CD} = \widehat{DA}$.	3. § 141.
4. $ABCD$ is a square.	4. § 294.

Investigation Problem. Can you inscribe in a circle any other regular polygon besides the square and octagon? If a regular hexagon were inscribed in a circle, how many degrees of arc would each side cut off? Can you construct a central angle which will intercept this arc? Measure off six of these arcs on the circle. Is there any arc left over? Prove that your construction will cut the circle into six equal arcs with no remainder.

Is the construction which you have just completed of use in inscribing an equilateral triangle in the circle?

PROPOSITION 14

297. *A regular hexagon can be inscribed in a circle.*



Given: $\odot O$.

To prove: A regular hexagon can be inscribed in $\odot O$.

Construction:

STATEMENTS	REASONS
1. With any point A on $\odot O$ as center and OA as radius, cut the circle at B.	1. Post. 3.
2. Similarly, with the same radius, and B, C, D, and E in succession as centers, cut the circle at C, D, E, and F.	2. Post. 3.
3. Draw AB, BC, CD, DE, EF, and FA. $ABCDEF$ is the required hexagon.	3. Post. 1.

Proof:

STATEMENTS	REASONS
1. Draw AO and BO.	1. Post. 1.
2. $AB = AO = BO$.	2. Const.
3. $\angle AOB = 60°$.	3. § 88.
4. $\widehat{AB} = 60°$.	4. § 162.
5. The circle is divided into 6 equal arcs.	5. $360 \div 60 = 6$.
6. $ABCDEF$ is a regular hexagon.	6. § 294.

298. Corollary. *An equilateral triangle can be inscribed in a circle.*

Class Exercises

1. A regular octagon can be inscribed in or circumscribed about a given circle.

2. A square can be circumscribed about a circle.

3. Regular polygons of 4, 8, 16, 32, . . . 2^n, sides can be circumscribed about any circle, or inscribed in any circle.

4. Regular polygons of 12, 24, . . . sides can be inscribed in, or circumscribed about, any circle.

5. A triangle is regular, if its inscribed and circumscribed circles have the same center. Is a quadrilateral necessarily regular, if its inscribed and circumscribed circles have the same center? Explain

6. The radius of the circle inscribed in an equilateral triangle equals one-half the radius of the circle circumscribed about the triangle.

7. The radius of the circle inscribed in an equilateral triangle equals one-third the altitude of the triangle.

8. The diagonal AD of regular hexagon $ABCDEF$ is perpendicular to the diagonal CE.

9. If a point moves about inside a regular polygon, the sum of the perpendiculars upon the sides is constant.

10. The area of an equilateral triangle inscribed in a circle is half that of the regular hexagon inscribed in the same circle.

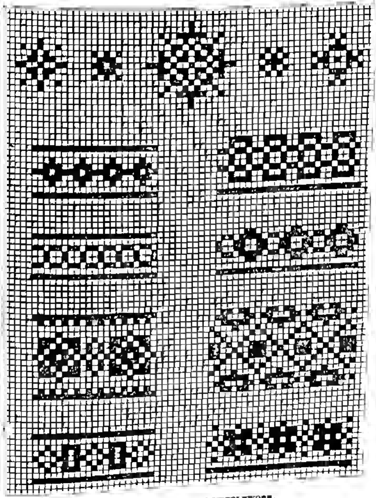
11. The square circumscribed about a circle equals twice the inscribed square.

12. The diagonals from a vertex of a regular polygon of n sides divide the angle into $(n-2)$ equal angles.

13. A central angle of a regular polygon is the supplement of an angle of the polygon.

14. If two diagonals of a regular polygon intersect, the product of the segments of one equals the product of the segments of the other.

15. The perimeter of an inscribed equilateral triangle is half that of the equilateral triangle circumscribed about the same circle.



USE OF THE SQUARE IN NEEDLEWORK.

Many beautiful embroidery designs are based on the square as a motive

Optional Exercises

16. The radius of an inscribed regular polygon is a mean proportional between the radius of its inscribed circle and the radius of the circumscribed regular polygon having the same number of sides.

17. If squares are constructed on the sides of a regular hexagon outside the hexagon, the twelve exterior vertices of these squares are the vertices of a regular dodecagon (12 sides).

18. The area of a square inscribed in a semicircle is two-fifths of the area of the square inscribed in the circle.

19. The diagonals of a regular pentagon are equal.

20. Find the area of the square inscribed in a circle whose radius is 8.

21. Find the area of a regular hexagon whose side is 10.

22. Find the side of an equilateral triangle equal in area to a regular hexagon whose side is 12.

23. Find the area of a regular hexagon whose radius is r.

24. Find the area of a regular hexagon circumscribed about a circle whose radius is 10.

25. In a circle whose radius is 12, find a side of the inscribed equilateral triangle.

26. Find the area of the equilateral triangle circumscribed about a circle whose radius is 12.

Honor Work

27. Find the side of a regular octagon inscribed in a circle whose radius is 10.

28. A square, each of whose sides is 8, has its corners cut off so as to make it a regular octagon. Find the area of the octagon.

29. A regular hexagon is reduced to one of smaller size by cutting straight across from the middle point of each side to that of the next. What proportion of the area is removed?

30. An equiangular polygon, circumscribed about a circle, is equilateral.

31. Construct a circumference equal to the sum of two given circumferences; one equal to their difference.

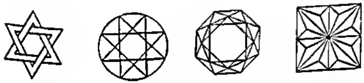
32. Construct a circle whose area equals the sum of the areas of two given circles; one equal to their difference.

33. Bisect the area of a circle by another circle having the same center.

34. On a given line as a side, construct a regular hexagon; a regular octagon.

Drawing Exercises

35. Copy the following figures. Try to discover other artistic designs.



A Self-Measuring Test

1. If two polygons are similar, what is the relation of their sides to their areas? to their perimeters?

2. On what proposition does the formula for the area of a circle depend?

3. Can a circle be circumscribed about:

(a) Any triangle?
(b) Any quadrilateral?
(c) Any polygon?
(d) An equilateral triangle?
(e) A rhombus?
(f) A regular hexagon?
(g) A square?
(h) A right triangle?

4. Is the proposition "The area of a regular polygon equals half the product of its perimeter and the radius of its inscribed circle" equally applicable to a polygon which is not regular (a) in any case? (b) in case a circle can be inscribed in the polygon? Can the area of an equilateral triangle be found by a similar method? the area of any triangle?

5. State the proposition about the sum of the angles of a triangle: of a polygon.

6. In proving that similar polygons can be cut into similar triangles, which of the propositions about similar triangles is used?

7. Give the formula for:

(a) The circumference of a circle.
(b) The area of a circle.
(c) The area of a trapezoid.
(d) The area of a triangle.
(e) The sum of the angles of a polygon.
(f) An angle of a regular polygon.

8. If you knew the area and the sides of a regular polygon, hor could you determine the radius of the inscribed circle?

9. State five propositions whose converses are true.

10. Explain the meaning of the statement that "an angle i measured by an arc."

11. What is a *mean proportional?* *A fourth proportional?*

12. Give at least one later proposition the proof of which depend on the fact that:

(a) The base angles of an isosceles triangle are equal.
(b) Alternate interior angles of parallel lines are equal.
(c) An exterior angle of a triangle is greater than a remote interior angle.
(d) If three or more parallels cut equal lengths . . . .
(e) The sum of the angles of a triangle equals . . . .
(f) A central angle is measured by its arc.
(g) A tangent is perpendicular to the radius . . . .
(h) An inscribed angle is measured by . . . .
(i) Triangles are similar, if two angles of one equal . . . .
(j) If an altitude is drawn on the hypotenuse of a right triangle, a leg is the mean proportional . . . .
(k) Right triangles are congruent, if the hypotenuse and leg . . . .
(l) Triangles are similar, if two sides are proportional and the included angles are equal.
(m) Similar triangles are to each other as the squares of corresponding sides.
(n) The area of the parallelogram equals . . . .
(o) Triangles are congruent, if two sides and the included

anglo of one equal .... If two angles and the included side of one equal ....

(p) The area of a triangle equals ....

13. State five propositions in whose proof we move the figure.

14. State a proposition which we prove by the indirect method.

15. Give two important propositions about loci.

16. Give three methods of proving an angle a right angle.

17. Give two propositions true for right triangles but not generally true for other triangles.

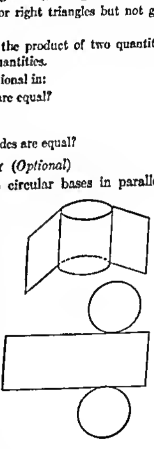
18. Give the method of proving the product of two quantities equal to the product of two other quantities.

19. To what are the areas proportional in:

(a) Triangles whose bases are equal?
(b) Similar triangles?
(c) Rectangles?
(d) Rectangles whose altitudes are equal?

SPACE GEOMETRY *(Optional)*

The circular cylinder has two circular bases in parallel planes, and a curved lateral surface. The lateral surface could be traced by a straight line always intersecting one of the circles and remaining parallel to its first position. The only cylinders considered here are those in which the moving line is perpendicular to the plane of the bases.



We can imagine a cylinder to be a prism whose base has a very great number of sides and from this we can infer theorems about the cylinder just as we assumed § 286 and § 292 for the circle from theorems about regular polygons.

Exercises

1. If a paper cylinder is cut open along a straight line and flattened out, the lateral surface becomes a rectangle. What line in the cylinder becomes the base of the rectangle? What line becomes its altitude?

2. Find the lateral area of a cylinder if the altitude is 10 and the circumference of its base is 25.

3. Find the lateral area of a cylinder if its altitude is 6 and the radius of its base is 4. Also find its total area.

4. Since the cylinder can be thought of as a special kind of prism, its volume will evidently equal the product of its base and altitude. If the altitude is 5 and the radius 6, find the volume.

5. Make a formula for the volume of a cylinder in terms of the altitude h and the radius of its base r.

6. A cylindrical tin can is to have a radius of 2 in. and a height of 3 in.

(a) How many square inches of tin are needed to make it?
(b) How many cubic inches of salmon will it hold?

7. If a gallon of paint will cover 400 sq. ft., how many gallons will be needed to paint the lateral surface of a cylindrical gas tank whose height is 30 ft. and whose radius is 20 ft.? How many cubic feet of gas will this tank hold?

8. In making a cylindrical measuring can whose radius is 2 in., how far from the bottom should the mark be placed to indicate 1 qt. if a quart contains 58 cu. in.?

9. A roll of ribbon has an average radius of 2 in. and contains 30 turns of ribbon. How many yards does it contain?

Review Exercises: Honor Work

1. Construct a circle equal to one-third of a given circle.

2. Find the length of an arc of 40°, if the radius is 10.

3. Find the number of degrees in an arc equal in length to the radius.

4. If the perimeter is 60 in each case, find the area of (a) an equilateral triangle; (b) a square; (c) a regular hexagon; (d) a circle. What conclusion would you draw from the results obtained?

5. If the area is 100 in each case, find the perimeter of (*a*) an equilateral triangle; (*b*) a square; (*c*) a regular hexagon; (*d*) a circle. What conclusion would you drew from the results obtained?

6. Divide a circle into arcs in the ratio $3:4:5$.

7. If the area of an equilateral triangle is $36\sqrt{3}$, find the radius of its inscribed circle; of its circumscribed circle.

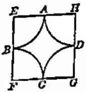
8. If the area of a regular hexagon is $96\sqrt{3}$, find the area of its inscribed circle.

9. In a circle whose radius is 10, find the area of the segment cut off by a chord equal to the radius.

10. Find the area of the largest circle which can be cut from a square of paper 10 in. on a side.

11. Construct a semicircle whose area will equal that of a given circle.

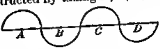
12. The diameters of two wheels are 20 in. and 28 in. The smaller makes 100 more revolutions in going a certain distance than the larger does. Find the distance.



13. If the vertices of a square 10 in. on a side are taken as the centers of arcs whose radii are 5, as in the figure, find the length of the curve $ABCD$.

14. Using the same hypothesis, find the area inclosed by the arcs.

15. The wave in the figure is constructed by taking A, B, C, and D as centers, and drawing semicircles with a radius of 2. Find the length of the curve.



Applied Problems

16. A square barn is 30 ft. on a side. A cow is tied at one corner by a rope 40 ft. long. Over how many square feet of ground can she graze?

17. A force of 100 lbs. is exerted on the small piston of a hydraulic press. If this piston has a radius of 1 in., what is the pressure per square inch on the liquid inside?

18. In Ex. 17, if the same pressure per square inch is transmitted to the large piston, the radius of which is 14 in., what is the total force exerted on this piston?

19. In order to construct a circular running track 100 yds. long, an athletic club purchases a square field. Find the length of the smallest field that could be used.

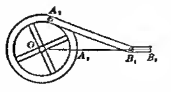
20. A circular running track is 20 ft. wide. How many feet start should be given to a boy running on the outside edge so that in going around the track once he will run the same distance as a boy running on the inner edge?

21. The amount of work done in turning a crank is the product of the force exerted by the distanre through which the force acts. How many foot-pounds of work are done in turning a crank completely around 20 times, if its radius is $1\frac{1}{3}$ ft. and if the force applied is 50 lbs.?

22. A shaft is turning at the rate of 60 revolutions per minute, and carries pulleys 6 in. in diameter. If the pulley on a machine is connected with one of these pulleys by a belt, what diameter must it have in order that it may make 40 revolutions per minute?

23. If in the above case, the pulley on the machine had a diameter of 5 in., how many revolutions per minute would it make?

24. Wheel O, which has a radius of 8, is turning, and rod AB, whose length is 17, is connected to the wheel at A. As A moves from the position A_1 on OB to the position A_2, where $OA_2 \perp A_2B_2$, find the distance B moves (B_2B_1).



25. Find the length of the belt which connects two equal pulleys, which have radii of 3 in., if their centers are 18 in. apart.

26. Find the length of the belt which connects two unequal pulleys, radii 7 in. and 2 in., if their centers are 10 in. apart.

27. A circular ring is 2 in. wide and has an area of 32 sq. in Find the inner radius.

28. How many feet of copper wire are required for the coils of a radio receiver, if the primary has 8 turns and the secondary 65

turns, both wound on a 3-in. cylinder, and the tickler has 40 turns on a 2-in. cylinder? (Assume each turn a circle.)

29. If a foot of a certain wire has a resistance of one ohm, how many turns must be wound on a 2-in. cylinder to produce a resistance of 2,000 ohms?

30. The resistance of a wire varies inversely as the area of a cross section, that is, a wire having twice as great a cross section has half as great a resistance. It is also directly proportional to the length. What length of wire $\frac{3}{16}$ in. in diameter will be required to produce the same resistance as a wire $\frac{1}{32}$ in. in diameter and 30 ft long?

31. If the sun is 390 times as far from the earth as is the moon, but appears to be the same size as the moon, how many times as great surface has it? How many times as great is its diameter? (Consider the disks of the sun and moon to be circles.)

INEQUALITIES

In most of the theorems so far studied, we proved things equal. It is however often as important to know when things are unequal as when they are equal. So now we shall study inequalities. These theorems have certain new values for us. In connection with the corresponding equality theorems, they enable us to trace the change in one quantity due to the change in another. And they provide a new form of the indirect proof in which we deal with more than two possibilities, and consequently must prove more than one of them wrong before we can establish the one that we wish to prove. This is important, for in life we often deal with situations in which there are more than two possibilities.

INEQUALITIES

299. Axioms.

15. If equals are added to unequals, the results are unequal in the same order.

16. If equals are subtracted from unequals, the results are unequal in the same order.

17. If unequals are multiplied by equals, the results are unequal in the same order.

18. If unequals are divided by equals, the results are unequal in the same order.

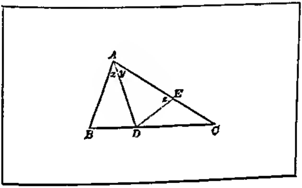
19. If unequals are subtracted from equals, the results are unequal in the opposite order.

20. A quantity must be less than, equal to, or greater than, another quantity of the same kind.

21. If the first of three quantities is greater than the second, and the second is greater than the third, the first is greater than the third.

Proposition 1

300. *If two sides of a triangle are unequal, the angle opposite the longer side is greater than the angle opposite the shorter side.*



Given: $[\triangle ABC]$; $AC > AB$.

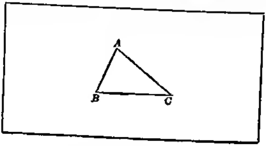
To prove: $\angle B > \angle C$.

Proof:

Statements	Reasons
1. Const. AD bisecting $\angle A$.	1. § 65.
2. On AC, take $AE = AB$.	2. Post. 3.
3. Draw DE.	3. Post. 1.
4. In $\triangle ABD$ and ADE, $\angle x = \angle y$.	4. Const.
5. $AD = AD$.	5. Iden.
6. $AB = AE$.	6. Const.
7. $\triangle ABD \cong \triangle ADE$.	7. § 50.
8. $\angle B = \angle z$.	8. § 22.
9. $\angle z > \angle C$.	9. § 63.
10. $\angle B > \angle C$.	10. Ax. 1.

Proposition 2

301. *If two angles of a triangle are unequal, the side opposite the greater angle is longer than the side opposite the smaller angle.*



Given: [$\triangle ABC$]; $\angle B > \angle C$.

To prove: $AC > AB$.

Proof:

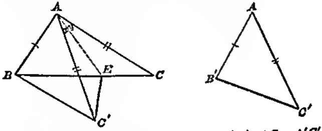
Statements	Reasons
1. $AC <, =,$ or $> AB$.	1. Ax. 20.
2. Suppose $AC < AB$. Then $\angle B < \angle C$.	2. § 300.
3. This is impossible because $\angle B > \angle C$.	3. Hyp.
4. Suppose $AC = AB$. Then $\angle B = \angle C$.	4. § 55.
5. This is impossible because $\angle B > \angle C$.	5. Hyp.
6. Therefore $AC > AB$.	6. The only possibility left.

302. **Corollary.** *The perpendicular is the shortest line from a point to a line.*

Draw any other line from the point to the line, and show that the side opposite the right angle is longer than the perpendicular.

Proposition 3

303. *If two triangles have two sides of one equal respectively to two sides of the other, and the included angle of the first greater than the included angle of the second, the third side of the first is longer than the third side of the second.*



Given: [$\triangle ABC$ and $A'B'C'$]; $AB = A'B'$, $AC = A'C'$, and $\angle A > \angle A'$.

To prove: $BC > B'C'$.

Proof: Place $\triangle A'B'C'$ so that $A'B'$ coincides with its equal AB. $A'C'$ falls inside $\angle BAC$ ($\angle BAC > \angle A'$ by hyp.). Construct AE bisecting $\angle C'AC$. Draw EC'. $AE = AE$ (Iden.). $AC' = AC$ (Hyp.). $\angle x = \angle y$ (Const.). $\triangle AEC' \cong \triangle AEC$ (§ 50). $EC' = EC$ (§ 22). $BE + EC' > BC'$ (Ax. 9). $BE + EC > BC'$ or $BC > B'C'$ (Ax. 1).

304. Converse. *If two triangles have two sides of one equal respectively to two sides of the other, and the third side of the first longer than the third side of the second, the angle opposite the third side of the first is greater than the angle opposite the third side of the second.*

Given: [$\triangle ABC$ and $A'B'C'$]; $AB = A'B'$, $AC = A'C'$, and $BC > B'C'$

To prove: $\angle A > \angle A'$.

Proof: $\angle A <, =,$ or $> \angle A'$ (Ax. 20). If $\angle A < \angle A'$, $BC < B'C'$ (§ 303); and if $\angle A = \angle A'$, $\triangle ABC \cong \triangle A'B'C'$ (§ 50) and $BC = B'C'$ (§ 22). But both of these are impossible, as $BC > B'C'$ (Hyp.). Therefore, $\angle A > \angle A'$, the only possibility left.

Exercises

1. If two sides of a triangle remain constant in length, but their included angle increases in size, what change takes place in the third side?

2. If, in $\triangle ABC$, $AB=12$ and $AC=16$, find the smallest and the largest values which the side BC can take as $\angle A$ increases from 0° to 180°. When $\angle A$ is obtuse, between what numbers must the length of BC lie? Has BC a different length for every different size of $\angle A$?

3. As $\angle A$ increases from 0° to 180°, while AB and AC remain constant in length, does the area of $\triangle ABC$ increase continuously? When $\angle A$ becomes nearly 180°, what can you say about the area of $\triangle ABC$?

4. If $\angle A$ doubles its size, does side BC double? Does the area of $\triangle ABC$ double?

5. Two sides of a triangle are hinged rods and the third is a stretched rubber band. If the angle included by the rods grows larger, what is the effect on the rubber band?

6. ABC is a triangle in which side AB is greater than side AC. OB and OC bisect angles B and C respectively. Prove that OB is greater than OC.

7. In each of the following triangles, determine whether $\angle C$ is acute, right or obtuse, and prove your conclusion by comparing the given value of c with that of the hypotenuse of a right triangle having the legs equal to the values of a and b given:

(a) $a=6,\ b=8,\ c=7$.
(b) $a=5,\ b=12,\ c=15$.
(c) $a=6,\ b=8,\ c=10$.
(d) $a=12, b=5, c=13$.
(e) $a=9, b=8, c=11$.
(f) $a=7, b=3, c=9$.

8. The sum of two sides of a triangle is greater than the third side.

9. The difference of two sides of a triangle is less than the third side.

10. Is it possible to construct a triangle whose sides are:

(a) 3, 4, 5.
(b) 1, 2, 3.
(c) 4, 6, 1.
(d) 5, 6, 7.

11. In $\triangle ABC$, $AB = AC$. Point D is on AB extended through B.
 (a) $\angle ACD > \angle D$. (c) $AD > AC$.
 (b) $DC > BC$. (d) $\angle ACB > \angle D$.

12. In $\triangle ABC$, $AB = AC$. Point D is on BC extended through C.
 (a) $AD > CD$. (b) $AD > AB$. (c) $\angle B > \angle D$.

13. A rectangle is not a square if a diagonal does not bisect the angles.

14. State and prove the converse of Ex. 13.

15. The sum of the diagonals of a quadrilateral is greater than the sum of a pair of opposite sides.

16. In ▱ $ABCD$, AC is the longer diagonal. P is a point on AC such that $AP = AB$. Then $BC > PC$.

17. If BC is the longest side and AD the shortest side of quadrilateral $ABCD$, then $\angle D > \angle B$. (Draw BD.)

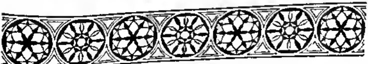
18. BC is the longest side of $\triangle ABC$ and D is any point on AC. Then $BC > BD$.

19. In an equilateral triangle, any line drawn from a vertex and ending in the opposite side is less than a side of the triangle.

20. In isosceles $\triangle ABC$, vertex $\angle A = 40°$. Then the angle bisector of a base angle is longer than either segment into which it divides the opposite side.

305. *In a circle, or in equal circles, if two central angles are unequal, the greater central angle intercepts the greater arc.* This follows immediately from § 162.

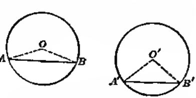
306. *In a circle, or in equal circles, if two arcs are unequal, the greater arc is intercepted by the greater central angle.* This follows immediately from § 162.



EACH UNIT OF THIS DESIGN IS SYMMETRICAL AS WELL AS THE DESIGN AS A WHOLE

Proposition 4

307. *In a circle, or in equal circles, the greater of two unequal arcs has the greater chord.*[1]



Given: $\odot O = \odot O'$, and $\widehat{AB} > \widehat{A'B'}$.
To prove: Chord $AB >$ chord $A'B'$.

Proof: Draw OA, OB, $O'A'$, and $O'B'$. $OA = O'A'$ (§ 138). $OB = O'B'$ (§ 138). $\angle AOB > \angle A'O'B'$ (§ 306). $AB > A'B'$ (§ 303).

308. **Converse.** *In a circle, or in equal circles, the greater of two unequal chords has the greater arc.*

Given: $\odot O = \odot O'$, and chord $AB >$ chord $A'B'$.
To prove: $\widehat{AB} > \widehat{A'B'}$.

Proof: Draw $OA, OB, O'A'$, and $O'B'$. $OA = O'A'$ and $OB = O'B'$ (§ 138). $AB > A'B'$ (Hyp.). $\angle O > \angle O'$ (§ 304). $\widehat{AB} > \widehat{A'B'}$ (§ 305).

Exercises

1. If arc AB is twice as long as arc $A'B'$, in equal circles, is chord AB twice, more than twice, or less than twice, chord $A'B'$? Prove your answer correct.

2. Prove that an arc of 180° has a chord twice as long as has an arc of 60°.

3. Have arcs the same ratio as their central angles? Have chords the same ratio as their central angles?

4. Is Proposition 3 still true, if both arcs are major arcs? If not, change its wording so that it will be true for this case only.

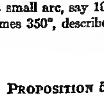
[1] An arc always means a minor arc, unless otherwise specified.

5. Could equal chords have unequal arcs if one of the arcs is a major arc?

6. If we start with a small arc, say 10°, and gradually increase its length until it becomes 350°, describe the changes which take place in its chord.

Proposition 5

309. *In a circle, or in equal circles, if two chords are unequally distant from the center, the chord nearer the center is the greater.*



Given: $\odot O$, $OG > OF$, $OF \perp AB$, and $OG \perp CD$.

To prove: $AB > CD$.

Proof: Construct $BE = CD$, and $OH \perp BE$. Draw FH. $OH = OG$ (§ 149). $OG > OF$ (Hyp.). $OH > OF$ (Ax. 1). $\angle w > \angle y$ (§ 300). $\angle OFB$ and $\angle OHB$ are rt. $\angle$s (§12). $\angle OFB = \angle OHB$ (§ 33). $\angle x < \angle z$ (Ax. 19). $FB > BH$ (§ 301). $AB = 2FB$ and $BE = 2BH$ (§ 147). $AB > BE$ (Ax. 17). $AB > CD$ (Ax. 1).

310. Converse. *In a circle, or in equal circles, if two chords are unequal, the greater is nearer the center.*

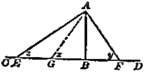
Given: $\odot O$, $AB > CD$, $OF \perp AB$, and $OG \perp CD$.

To prove: $OG > OF$.

Proof: $OG <$, $=$, or $> OF$ (Ax. 20). If $OG < OF$, $CD > AB$ (§ 309); and if $OG = OF$, $AB = CD$ (§ 150). Both of these are impossible, as $AB > CD$ (Hyp.). Therefore, $OG > OF$, the only possibility left.

Proposition 6

311. *If two lines are drawn from a point on a perpendicular to a line, cutting unequal lengths from the foot of the perpendicular, that cutting the greater length is the greater.*



Given: $AB \perp CD$, $EB > BF$.
To prove: $AE > AF$.

Proof: On EB take $GB = BF$, and draw AG. $AG = AF$ (§ 59). Point G is between E and B. ($EB > BF$ by hyp.). $\angle z = \angle y$ (§ 55). $\angle z > \angle x$ (§ 68). $\angle y > \angle x$ (Ax. 1). $AE > AF$ (§ 301).

312. Converse. *If two unequal lines are drawn from a point on a perpendicular to a line, the greater cuts off the greater length from the foot of the perpendicular.*

Given: $AB \perp CD$, $AE > AF$.
To prove: $EB > BF$.

Proof: $EB <, =$, or $> BF$ (Ax. 20). If $EB < BF$, $AE < AF$ (§ 311); and if $EB = BF$, $AE = AF$ (§ 59). Both of these are impossible, as $AE > AF$ (Hyp.). Therefore, $EB > BF$, the only possibility left.

Exercises

1. In the $\triangle ABC$, if $\angle A = 60°$ and $\angle B > \angle C$, which is the longest side of the triangle?

2. In quadrilateral $ABCD$, $AB = BC$, and $\angle A > \angle C$. Explain which is longer, AD or CD.

3. In quadrilateral $ABCD$. $AB > BC$, and $\angle A = \angle C$. Explain which is longer, AD or CD.

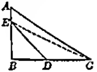
4. In $\triangle ABC$, $AB = AC$ and D is any point on BC. Prove that AB is longer than AD.

5. In $\triangle KLM$, $KM > KL$, and P is any point on LM. Prove that KM is longer than KP.

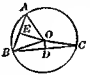
6. Prove section 308 by the indirect method.

7. For section 310 give a direct proof similar to that of 309.

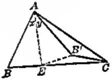
8. If a rectangle, not a square, is inscribed in a circle, the adjacent sides are unequally distant from the center.



Ex. 9.



Exs. 12, 13.



Ex. 15.

9. If $AB \perp BC$, prove that $AC > ED$.

10. In a circle, if one arc is twice another, the chord of the first is less than twice the chord of the second.

11. The diagonals of a rhombus, not a square, are unequal.

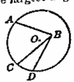
12. If in $\odot O$, $OE \perp AB$, $OD \perp BC$, and $OE > OD$, prove that $\angle BOC > \angle AOB$.

13. If $OE \perp AB$, $OD \perp BC$, and $\angle BOC > \angle AOB$, prove that $OE > OD$.

14. Show that, when boatds are sawed from a circular log, each board is wider than the one before it, until the center of the log is reached.

15. Prove the proposition of section 303, using the figure given above.

16. If the sides of a triangle are 8, 8, and 10, is the largest angle acute, right, or obtuse?



17. If $AB = BD$ and $\angle ABC > \angle CBD$, then $\widehat{AC} > \widehat{CD}$.

18. If $AB = BD$ and $\widehat{AC} > \widehat{CD}$, then $\angle ABC > \angle CBD$.

19. Show that $\triangle ABC$ is obtuse, if $AB = 9$, $BC = 12$, and $AC = 18$.

20. In a triangle, the square of the side opposite an acute angle is less than the sum of the squares of the other two sides.

21. If two triangles have a side and an adjoining obtuse angle of one equal to a side and an adjoining obtuse angle of the other, but the other including side of the first obtuse angle longer than the other including side of the second obtuse angle, then the third side of the first is longer than the third side of the second.

22. The hypotenuse of a right triangle is longer than either leg.

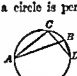
23. If two triangles have an obtuse angle of one equal to an angle of the other, but both including sides of one longer than the corresponding sides of the other, then the third side of the first is longer than the third side of the second.

24. In $\triangle ABC$, the bisectors of $\angle B$ and C meet at D. Prove that BC is longer than either BD or DC.

25. If, in $\triangle ABC$, $AB=AC$ and D is any point on AC, then $DB>DC$.

26. The shortest line from a point inside a circle to the circle is the part of the radius through the point.

27. The shortest chord through a point inside a circle is perpendicular to the radius through the point.



28. If AD is a median of $\triangle ABC$ and $AC>AB$, then $\angle ADC$ is obtuse.

29. If chord $AB>$ chord CD, then chord $AC>$ chord BD.

30. If chord $AC>$ chord BD, then chord $AB>$ chord CD.

31. If D, E, and F are the middle points respectively of the radii AO, BO, and CO of $\odot O$, and $DE>EF$, then $\widehat{AB}>\widehat{BC}$.

32. If two circles have the same center, and three radii cut off two unequal arcs on the larger circle, they will cut off unequal arcs on the smaller.

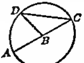
33. Quadrilateral $ABCD$ is jointed at A, B, C, and D. If $\angle B$ increases, prove that $\angle D$ also increases. If $\angle B$ reaches 180° before $\angle D$ does, prove that $\angle D$ cannot become

180°. Similarly, if $\angle B$ reaches 0° before $\angle D$ does, prove that $\angle D$ cannot become 0°.

34. In Proposition 6, if BF grows larger, what change takes place in AF? If BF doubles its length, does AF double? If BF decreases to 0, to what does AF decrease?

35. In Proposition 5, if one chord is one half as far from the center as the other, is it twice as long?

36. *The diameter of a circle is greater than any other chord.*



$$AC = DB + BC.$$
$$DB + BC > DC.$$

37. In $\square ABCD$, $\angle B$ is acute. Prove that diagonal BD is longer than diagonal AC. If AC increases in length while the sides remain constant, what change takes place in BD?

Geometric Reasoning Applied to Life Situations

38. A solution contains only one metal. The chemist wishes to find out whether it is lead, copper or iron. If he puts in acid no. 1, it will turn white if it contains lead. If he puts in acid no. 2, it will turn black if it contains either lead or copper. But if it contains iron, neither acid will affect it. He puts in acid no. 1 and no change takes place. He puts in acid no. 2 and it turns black. Decide which metal is in the solution and prove your conclusion.

39. I found a geometry book. George, Jacob, Ted and Hans are the only pupils who have lost theirs. This book is new and neatly covered, but has no name in it. George had a new book, covered it, and wrote his name in it. Jacob had a new book, did not write his name in it or cover it. Ted covered his book but forgot to write his name in it. Hans had an old book. He covered it but did not write his name in it. To whom shall I return the book? Prove your answer.

40. Consider the proposition "Breaking a mirror brings bad luck." If the following statements are true, does any one of them prove or disprove the proposition?

(a) Robert broke a mirror and afterwards lost his pocketbook.
(b) John did not break a mirror but he sprained his ankle.

(c) Dorothy did not break a mirror and had good luck.

(d) Paul broke a mirror and continued having good luck.

What is your own opinion about the truth of this proposition?

41. Either (a) all horses run fast, or (b) some horses run fast, or (c) no horses run fast. If Dobbin cannot run fast but Black Beauty can, which is true (a), (b), or (c)? Prove your answer.

42. You are a detective. Mrs. Brown was killed by a hit-and-run driver. Which, if any, of the following facts would be sufficient for proving Smith guilty?

(a) The radiator of his car was dented.

(b) Hair like Mrs. Brown's was found on the bumper.

(c) Smith drove home a short time after the tragedy.

(d) A neighbor saw Smith washing his car soon after he arrived home.

(e) An eye witness of the accident said that the car which hit Mrs. Brown was light blue. Smith's car was light blue.

(f) A check-up by the police showed that no other light blue car had been over that road at the time of the accident.

43. Charles wishes to call on Mary who lives in the three-story building, but he has forgotten on which floor she lives. He recalls that she once said that she went up stairs to lunch, and at another time said that Jane came down stairs to study with her. On what floor does Mary live?

44. Robert, Henry and James play in the school orchestra. One plays the violin, one the piano, and one the trumpet. Robert and Henry whistle while they are playing their instruments. Robert always carries his instrument along when he goes on a picnic. What instrument does Henry play? Prove your answer.

45. Mr. Smith and Mr. Jones each have a son. The sons' names are Paul and John. Mr. Jones earns $3400 a yr. Mr. Smith lives in Cleveland. Paul earns exactly one-third as much a year as his father. What is John's last name?

46. In using the indirect proof, you must be careful that the possibilities are mutually exclusive, that is, that it must be either one or the other and *not both*. Also you must be careful to name all of the possibilities. Examine these proofs to see if they really prove the assertion.

(a) Either you are stupid or you are careless. I find that you are careless. Therefore you are not stupid.

(b) Either Allan is taller than John or John is taller than Allan. Allan is not taller than John. Therefore John is taller than Allan.

(c) Either you must do your homework or you will fail. You do your homework. Therefore you will not fail.

(d) After Lloyd and Roy left my house, I noticed that my book was gone. Lloyd did not take it. Therefore Roy must have done so.

47. Assume that teachers always tell the truth but that lawyers always lie. Wilson and Blair are either teachers or lawyers. Wilson says that Blair said he was a lawyer. Is Wilson a teacher or a lawyer?

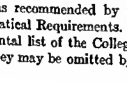
SUPPLEMENT

The following propositions, with the exception of Proposition 8, which is placed here because it logically belongs with the formula group, are not included in the list of fundamental propositions recommended by the National Committee on Mathematical Requirements. And none of them is on the fundamental list of the College Entrance Examination Board. They may be omitted by schools desiring a shorter course.

Lines Meeting in a Point

Proposition 1

313. *The bisectors of the angles of a triangle meet in a point equally distant from the sides of the triangle.*



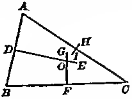
Given: [$\triangle ABC$]; AD, BE, and CF bisecting $\measuredangle A$, B, and C.

To prove: AD, BE and CF meet in a point O, equally distant from the sides.

Proof: $\angle A + \angle B < 1$ st. $\angle$ (§ 86). $\angle r + \angle s < 1$ st. $\angle$ (Ax. 18). AD and BE will meet in some point O (§ 83). Let x, y, and z be the distances from O to the sides of $\triangle ABC$. $x = y$ and $y = z$ (§ 97). $x = z$ (Ax. 2). CF will pass through O (§ 98).

PROPOSITION 2

314. *The perpendicular bisectors of the sides of a triangle meet in a point equally distant from the vertices.*



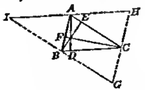
Given: [$\triangle ABC$]; $DE \perp$ bisector of AB, $FG \perp$ bisector of BC and $HI \perp$ bisector of AC.

To prove: DE, FG, and HI meet in a point O, equally distant from A, B, and C.

Proof: If FO were $\parallel DE$, $AB \perp FO$ (§ 81). Then AB would be $\parallel BC$ (§ 75). But AB and BC meet in B. Therefore DE meets FO in some point O. $OA = OB$ (§ 59). $OB = OC$ (§ 59). $OA = OC$ (Ax. 2). HI passes through O (§ 60).

PROPOSITION 3

315. *The altitudes of a triangle meet in a point.*



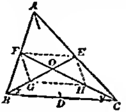
Given: [$\triangle ABC$]; altitudes AD, BE, and CF.

To prove: AD, BE and CF meet in a point.

Proof: Construct $HI \parallel BC$ through A, $HG \parallel AB$ through C, and $GI \parallel AC$ through B (§ 77). $IA = BC$ and $AH = BC$ (§ 105). $IA = AH$ (Ax. 2). $AD \perp IH$ (§ 81). Therefore AD is the $\perp$ bisector of IH. Similarly, BE and CF are $\perp$ bisectors of IG and HG respectively. Therefore AD, BE, and CF meet in a point (§ 314).

PROPOSITION 4

316. *The medians of a triangle meet in a point two-thirds of the distance from each vertex to the middle point of the opposite side.*



Given: [$\triangle ABC$]; medians AD, BE, and CF.

To prove: AD, BE, and CF meet in a point.

Proof: If BE and CF are $\parallel$, $\angle x+\angle y=180°$ (§ 83). But $\angle B+\angle C<180°$ (§ 86). This is impossible, for a part would be greater than the whole (Ax. 8). Therefore BE and CF meet in some point O. Bisect BO and CO at G and H (§ 63). Draw FE, EH, GH and FO (Post. 1). $FE \parallel BC$ and $=\frac{1}{2}BC$ and $GH \parallel BC$ and $=\frac{1}{2}BC$ (§ 221). $FE=GH$ (Ax. 6). $FE \parallel GH$ (§ 82). $FEHG$ is a $\square$ (§ 110). $GO=OE$ (§ 111). $BG=GO$ (Const.). $OE=\frac{1}{3}BE$. Similarly AD cuts off $\frac{1}{3}$ of BE. Therefore AD, BE, and CF meet at point O.

EXERCISES

1. Given $\triangle ABC$, construct $\triangle DEF$ so that the perpendicular bisectors of the sides of $\triangle ABC$ shall be altitudes of $\triangle DEF$

2. The bisector of an angle of a triangle and the bisectors of the two remote exterior angles meet in a point.

3. Base BC of $\triangle ABC$ is extended both ways, and the bisectors of the interior and exterior angles at B and C drawn. If the bisector of the interior angle at B meets the bisector of the exterior angle at C in the point K, and the bisector of the interior angle at C meets the bisector of the exterior angle at B in the point L, prove that K, A, and L are in a straight line.

4. The point of intersection of two altitudes of a triangle cannot bisect both altitudes.

5. If the perpendicular bisectors of three sides of a quadrilateral meet in a point, the perpendicular bisector of the fourth side passes through that point.

6. If the bisectors of three angles of a quadrilateral meet in a point, the bisector of the fourth angle passes through that point.

7. Construct a triangle, given a side and the medians to the other two sides.

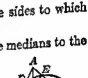
8. Construct a triangle, given two medians and the side to which one of them is drawn.

9. If the perpendicular bisectors of two sides of a triangle meet on the third side, the triangle is a right triangle.

10. Find a point equally distant from three sides of a parallelogram. Is this point necessarily the same distance from the fourth side? Explain.

11. If two medians of a triangle are equal, the sides to which they are drawn are equal.

12. Two triangles are congruent, if a side and the medians to the other two sides of one triangle equal a side and the corresponding medians of the other.



13. A circle, constructed on side BC of $\triangle ABC$ as a diameter, cuts the other two sides in D and E. If BE and CD intersect in F, prove that AF is perpendicular to BC.

14. The bisectors of all angles inscribed in a given segment of a circle pass through a common point.

15. In quadrilateral $ABCD$, the perpendicular bisectors of AB and AD meet at point E, and the perpendicular bisectors of BC and CD meet at point F. Then EF is the perpendicular bisector of BD.

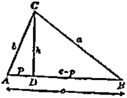
16. In an isosceles trapezoid, the perpendicular bisector of the base passes through the point of intersection of the non-parallel sides produced.

17. Three equal circles are tangent to each other. Prove that the three common tangents through their points of contact meet in a point which is equally distant from the centers of the circles.

FORMULA PROPOSITIONS

PROPOSITION 5

317. *If in $\triangle ABC$, A is an acute angle, $a^2=b^2+c^2-2cp$ where p is the distance from A to the foot of the altitude from C.*



Given: [$\triangle ABC$]; $\angle A$ acute, altitude CD and $AD=p$.

To prove: $a^2=b^2+c^2-2cp$.

Proof:
$$a^2=h^2+(c-p)^2 \text{ (§ 227)}$$
$$=h^2+p^2+c^2-2cp.$$

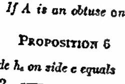
But $h^2+p^2=b^2$ (§ 227).

Therefore $a^2=b^2+c^2-2cp$ (Ax. 1).

318. Corollary. *If A is an obtuse angle, $a^2=b^2+c^2+2cp$.*

PROPOSITION 6

319. *The altitude h_c on side c equals*
$$\frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)},$$
where
$$s=\frac{a+b+c}{2}.$$



Given: [$\triangle ABC$]; altitude h_c.

To prove: $h_c=\frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$.

Proof:

$$a^2=b^2+c^2-2cp \quad (\S\ 317)$$
$$2cp=b^2+c^2-a^2$$
$$p=\frac{b^2+c^2-a^2}{2c}$$

Let $2s=a+b+c$

Then $2(s-a)=b+c-a$

$2(s-b)=a-b+c$

$2(s-c)=a+b-c$

$$h_c^2=b^2-p^2 \quad (\S\ 227)$$
$$=b^2-\left(\frac{b^2+c^2-a^2}{2c}\right)^2 \quad \text{(Ax. 1)}$$
$$=\left(b+\frac{b^2+c^2-a^2}{2c}\right)\left(b-\frac{b^2+c^2-a^2}{2c}\right)$$
$$=\frac{(b+c+a)(b+c-a)(a-b+c)(a+b-c)}{4c^2}$$
$$=\frac{16s(s-a)(s-b)(s-c)}{4c^2}$$
$$h_c=\frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

320. Corollary. *The area of a triangle equals*

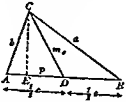
$$\sqrt{s(s-a)(s-b)(s-c)}.$$

$$K=\tfrac{1}{2}c\cdot h \quad (\S\ 251)$$
$$=\frac{1}{2}c\cdot\frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)} \quad \text{(Ax. 1)}$$
$$=\sqrt{s(s-a)(s-b)(s-c)}.$$

Proposition 7

321. *The median m_c to side c of $\triangle ABC$ equals*

$$\tfrac{1}{2}\sqrt{2a^2+2b^2-c^2}.$$



Given: [$\triangle ABC$ with] median $CD = m_c$.

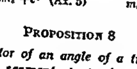
To prove: $m_c = \tfrac{1}{2}\sqrt{2a^2+2b^2-c^2}$.

Proof: Draw $CE \perp AB$. Using $AD = \tfrac{1}{2}c$ in § 317 and § 318,

$$b^2 = m_c^2 + \tfrac{1}{4}c^2 - cp$$
$$a^2 = m_c^2 + \tfrac{1}{4}c^2 + cp$$
$$a^2+b^2 = 2m_c^2 + \tfrac{1}{2}c^2 \quad \text{(Ax. 3)}$$
$$2a^2+2b^2 = 4m_c^2 + c^2 \quad \text{(Ax. 5)}$$
$$4m_c^2 = 2a^2+2b^2-c^2$$
$$2m_c = \sqrt{2a^2+2b^2-c^2}$$
$$m_c = \tfrac{1}{2}\sqrt{2a^2+2b^2-c^2}$$

Proposition 8

322. *The bisector of an angle of a triangle divides the opposite side into segments proportional to the other two sides.*



Given: [$\triangle ABC$], AD bisects $\angle BAC$.

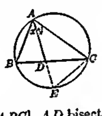
To prove: $\dfrac{BD}{DC} = \dfrac{BA}{AC}$.

Proof: Construct $BE \parallel AD$ (§ 77) and extend CA to E (Post. 2). $\frac{BD}{DC}=\frac{EA}{AC}$ (§ 208). $\angle w=\angle x$ (Hyp.). $\angle w=\angle z$ (§ 80). $\angle x=\angle y$ (§ 79). $\angle y=\angle z$ (Ax. 2). $EA=BA$ (§ 91). $\frac{BD}{DC}=\frac{BA}{AC}$ (Ax. 1).

323. Corollary. $BD=\frac{ac}{b+c}$ and $DC=\frac{ab}{b+c}$.

Proposition 9

324. *The square of the bisector of an angle of a triangle equals the product of the two including sides diminished by the product of the segments of the third side.*



Given: $[\triangle ABC]$, AD bisects $\angle A$.

To prove: $\overline{AD}^2=AB\times AC-BD\times DC$.

Proof: Circumscribe a $\odot$ about $\triangle ABC$ (§ 174). Extend AD to E and draw EC (Post. 1 and 2). $\angle B=\angle E$ (§ 167). $\angle x=\angle y$ (Hyp.). $\triangle ABD \sim \triangle AEC$ (§ 216). $AB : AD=AE : AC$ (§ 215). $AB\times AC=AD\times AE$ (§ 200). $AE=AD+DE$ (Ax. 7). $AB\times AC=AD(AD+DE)=\overline{AD}^2+AD\times DE$ (Ax. 1). $\overline{AD}^2=AB\times AC-AD\times DE$ (Ax. 4). $AD\times DE=BD\times DC$ (§ 230). $\overline{AD}^2=AB\times AC-BD\times DC$ (Ax. 1).

325. Corollary. $t_a^2=bc-\frac{ac}{b+c}\cdot\frac{ab}{b+c}$. (§§ 324 and 323)

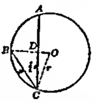
$$=\frac{(b+c)^2\,bc-a^2bc}{(b+c)^2}=\frac{bc[(b+c)^2-a^2]}{(b+c)^2}=\frac{4bcs(s-a)}{(b+c)^2},$$

$$t_a=\frac{2}{b+c}\sqrt{bcs(s-a)}.$$

PROPOSITION 10

326. *Given the side s and the radius r of a regular inscribed polygon, the side x of a regular inscribed polygon having double the number of sides is*

$$x=\sqrt{2r^2-r\sqrt{4r^2-s^2}}.$$



Given: $AC=s$, the side of a regular inscribed polygon. $BC=x$ the side of a regular inscribed polygon having double the number of sides.

To prove: $x=\sqrt{2r^2-r\sqrt{4r^2-s^2}}$.

Proof: Draw a radius perpendicular to AC. It will pass through B (§ 147).

Draw $OC=r$.

$$OD=\sqrt{r^2-\tfrac{1}{4}s^2} \qquad (\S\ 228)$$
$$BD=r-OD=r-\sqrt{r^2-\tfrac{1}{4}s^2} \qquad (\text{Ax. } 4)$$
$$x=\sqrt{\overline{DC}^2+\overline{BD}^2} \qquad (\S 227)$$
$$=\sqrt{\tfrac{1}{4}s^2+(r-\sqrt{r^2-\tfrac{1}{4}s^2})^2} \quad (\text{Ax. } 1)$$
$$=\sqrt{\tfrac{1}{4}s^2+r^2-2r\sqrt{r^2-\tfrac{1}{4}s^2}+r^2-\tfrac{1}{4}s^2}$$
$$=\sqrt{2r^2-2r\sqrt{r^2-\tfrac{1}{4}s^2}}$$
$$=\sqrt{2r^2-r\sqrt{4r^2-s^2}}$$

327. Corollary. *The value of π is approximately* 3.1416.

Proof: Let s_6 be the side of a regular inscribed polygon of 6 sides, s_{12} of one of 12 sides, etc. Then

$$s_{12}=\sqrt{2r^2-r\sqrt{4r^2-s_6^2}}\ (\S\ 326).$$

Now let r equal 1. Then s_6 will equal 1 (§ 297).

	Length of Side	Perimeter / Diameter
$s_{12}=\sqrt{2-\sqrt{4-1}}$	.517638	3.105829
$s_{24}=\sqrt{2-\sqrt{4-.517638^2}}$	.261052	3.132629
$s_{48}=\sqrt{2-\sqrt{4-.261052^2}}$	.130806	3.139350
$s_{96}=\sqrt{2-\sqrt{4-.130806^2}}$	.065438	3.141032
$s_{192}=\sqrt{2-\sqrt{4-.065438^2}}$	.032723	3.141452
$s_{384}=\sqrt{2-\sqrt{4-.032723^2}}$	.016362	3.141558
$s_{768}=\sqrt{2-\sqrt{4-.016362^2}}$	.008181	3.141585

Therefore, calculated to four decimal places, the value of π is 3.1416.

EXERCISES

1. The sides of a triangle are 6, 8, and 10. (*a*) Find the bisector of the largest angle. (*b*) Determine whether the largest angle is right, obtuse, or acute. (*c*) Find the altitude from the smallest angle. (*d*) Find the median to side 8. (*e*) Find the area of the triangle.

2. The legs of a right triangle are 9 and 12. Find the segments of the hypotenuse made by the bisector of the right angle.

3. In $\triangle ABC$, AB is 14 and BC is 12. If the bisector of $\angle C$ meets AB in D, and AD is 6, find AC.

4. In a right triangle, one of whose angles is 30°, the side opposite the 30° angle is 12. Find the median to this side.

5. Find the area of a rhombus, if the sides are each 10, and one diagonal is 12.

6. If two sides of a parallelogram are 6 and 8, and one diagonal is 7, find the other diagonal.

7. If two sides of a triangle are 8 and 10, but their included angle increases gradually from 0° to 180°, how does the third side change? What is its smallest length, and what is its greatest length?

8. Find the area of a triangle whose sides are: 6, 8, 2; 6, 8, 5; 6, 8, 10; 6, 8, 12; 6, 8, 14. How does the area change as the third side of the triangle increases from 2 to 14? For which of the values given is the area the largest? Is the angle between sides 6 and 8 acute, right, or obtuse for this value of the third side?

9. In a triangle whose sides are 8, 10, and 12, find the altitude, the median and the angle bisector to side 12. Which is the shortest, and which the longest? Can you change a side of the triangle so that the median and the angle bisector to side 12 will become the same length?

10. In the formula $a^2=b^2+c^2-2cp$ (§ 317), what change takes place in p as $\angle A$ increases in size? When $\angle A=60°$, how does p compare with b? As $\angle A$ becomes 90°, what value does p take? What can you say of p, when $\angle A$ becomes greater than 90°?

11. The sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the sides.

12. In each of the following triangles, find the length of the median to the longest side:

(a) $a=3, b=4, c=5$. (c) $a=6, b=6, c=8$.
(b) $a=7, b=8, c=9$. (d) $a=6k, b=8k, c=10k$.

13. In Ex. 12, find the angle bisector of the largest angle.

14. In Ex. 12, find the altitude to the shortest side.

15. In Ex. 12, determine, in each triangle, whether the largest angle is acute, right, or obtuse.

16. In Ex. 12, find the area of each triangle.

17. Find the length of the common chord of two circles, whose radii are 9 and 11, and whose centers are 14 apart.

18. Find the area of a parallelogram whose sides are 10 and 12, and whose diagonal is 14.

19. Find the area of a quadrilateral $ABCD$, if $AB=5$, $BC=9$, $CD=8$, $DA=10$, and $AC=12$.

20. Find the area of quadrilateral $ABCD$, if $AB=12$, $BC=10$, $CD=14$, $DA=16$, and $\angle A=90°$.

21. The sides of a triangle are 8, 15, and 17. Find the segment of the longest side intercepted between (*a*) the altitude on it and the bisector of the opposite angle; (*b*) the median to it and the bisector of the opposite angle.

22. A triangular park has 10 rods frontage on one street, 14 rods on the second, and 16 rods on the third. Find its area in square rods.

23. Find the area of a regular octagon whose radius is 6.

328. Summary of formulas.

AREAS

Rectangle	$K=ab$	§ 246
Parallelogram	$K=hb$	§ 250
	$K=ab \sin C$	§ 253
Triangle	$K=\frac{1}{2}hb$	§ 254
	$K=\frac{1}{2}ab \sin C$	§ 258
Trapezoid	$K=\frac{1}{2}h(b+b')$	§ 260
Regular Polygon	$K=\frac{1}{2}rp$	§ 290
Circle	$K=\frac{1}{2}rc$	§ 291
	$K=\pi r^2$	§ 292

ANGLES	MEASURED BY	
Central	its arc	§ 162
Inscribed	$\frac{1}{2}$ its arc	§ 165
Formed by tangent and chord	$\frac{1}{2}$ its arc	§ 170
Inside the circle	$\frac{1}{2}$ sum of its arcs	§ 171
Outside circle	$\frac{1}{2}$ difference of its arcs	§ 172

OTHER FORMULAS

Circumference	$c=2\pi r$	§ 283
In similar polygons	$\frac{a}{a'}=\frac{b}{b'}=\frac{p}{p'}$	§ 270
In circles	$\frac{c}{c'}=\frac{r}{r'}$	§ 286
Pi	$\pi=3.1416$	§ 327
Sum of angles of polygon	$(n-2)$ st. $\measuredangle$	§ 266
Sum of ext. $\measuredangle$ of polygon	2 st. $\measuredangle$	§ 267
Each angle of equiangular polygon	$\frac{n-2}{n}$ st. $\measuredangle$	§ 268
Central $\angle$ of regular polygon	$\frac{360}{n}$ degrees	§ 281
Side of $\triangle$ opposite acute $\angle$	$a^2=b^2+c^2-2cp$	§ 317
Side of $\triangle$ opposite obtuse $\angle$	$a^2=b^2+c^2+2cp$	§ 318
Side of $\triangle$ opposite right $\angle$	$c^2=a^2+b^2$	§ 227
Median of triangle	$m_c=\frac{1}{2}\sqrt{2a^2+2b^2-c^2}$	§ 321
Bisector of an angle of a triangle	$t_a=\frac{2}{b+c}\sqrt{bcs(s-a)}$	§ 323
Altitude of triangle	$h_c=\frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$	§ 319
Area of triangle	$K=\sqrt{s(s-a)(s-b)(s-c)}$	§ 320

THE STORY OF GEOMETRY[1]

First Episode: Thales

You have read in the introduction to this book how the Egyptians by necessity worked out a crude system of geometry that enabled them to relocate the boundaries of their farms after the Nile floods had obliterated the landmarks. Egypt, therefore, may be called the birthplace of geometry. But the Egyptians were not interested in proving that their rules were true, and they did not think of geometry as dealing with figures represented on paper as we do. To them, a point was a location or position on the earth, a line was a string or pole used to measure, and a surface was an area of land. They merely considered geometry as a practical art, an aid to surveying.

When information on any subject is collected and organized, it is called a science, and the first step toward making geometry a science was taken by Thales of Miletus, a Greek, who, about 600 B.C., transplanted Egyptian ideas of measuring to Greece. Therefore we think of geometry as a product of western civilization.

Thales was a trader in his younger days and had amassed great wealth, which enabled him to indulge his taste for learning, and to found the Ionian school of philosophy. Here he developed and

[1] To the teacher. We are forced to admit that the history of mathematics has not functioned as far as pupils are concerned. The splendid works of Smith, Karpinski, Cajori, Ball, and others are gold mines, but the ore is practically untouched. This text initiates a new line of attack through which it is hoped pupils will be stimulated to write and to read to each other compositions built around interesting episodes. It is always challenging to a pupil to learn what others have done. Hence in this geometry, the story of geometry unfolds as a serial consisting of episodes. These are based on compositions written by pupils of Raleigh Schorling, to whom we owe this treatment of material from the history of mathematics and which is included here with Professor Schorling's permission.

Eighth Episode: Later Writers

The Romans did not add to the science of geometry as the Greeks had done. They were a practical people and were interested in geometry only for its use in surveying, laying out roads, and in the engineering of warfare. The Hindus likewise obtained their basis of geometry from Egypt, but they too gave more attention to computation than to underlying principles. For a thousand years only those countries under Greek influence knew demonstrative geometry. The Arabs recognized Greek culture, and through them geometry was later introduced into Western Europe. It was not, however, until the fourteenth century that geometry became an essential item in university courses.

During the last centuries, many great mathematicians, such as Descartes, Newton, and Gauss, have extended the field of geometry, but it was not until Legendre wrote his text on geometry that the subject was simplified so that it could be used in schools lower than the university.

APPENDIX A

EXAMINATION REQUIREMENTS OF THE UNIVERSITY OF THE STATE OF NEW YORK

The courses of study in plane geometry in most of the states that prescribe courses agree in general with that recommended by the National Committee on Mathematical Requirements and with that of the College Entrance Examination Board. The syllabus of the State Department of Education of the University of the State of New York may be considered representative of state courses in general. For this reason the requirements of that syllabus are given below. Much of the wording used is quoted directly from that syllabus.

The examination will be made up of questions on book propositions and on originals. The former type of questions will require the demonstration of certain theorems or the reproduction of certain constructions which it is assumed have been presented in class. The policy in recent examinations has been to include one or at most two of these. The latter type of question, the originals, will include the solution of construction problems, loci problems, and numerical problems as well as the demonstration of theorems, all of which are assumed to be unfamiliar to the pupil.

The book propositions chosen for reproduction on the examination will be selected not from the complete list of propositions in this book but from the restricted list given

below. Although the pupil is not expected to have the proofs immediately in mind for the other propositions in this book, it is expected that he will be familiar with their content and that he will be able to use them in the solution of originals. Furthermore, some of these propositions may be given as originals on the examination if they seem to lend themselves readily to this purpose.

The list of propositions selected for reproduction on the examination with their paragraph numbers as given in this book is as follows:

55. The base angles of an isosceles triangle are equal (page 54).

57. Two triangles are congruent if the three sides of one equal the three sides of the other (page 58).

72. Two lines are parallel if their alternate interior angles are equal (page 87).

79. Alternate interior angles of parallel lines are equal (page 96).

86. The sum of the angles of a triangle is a straight angle (page 107).

91. If two angles of a triangle are equal, the sides opposite them are equal (page 114).

94. Two right triangles are congruent, if the hypotenuse and a leg of one equal the hypotenuse and a leg of the other (page 118).

104. The opposite sides of a parallelogram are equal (page 127). (This includes the proof of both the proposition 104 and the corollary 105.)

109. A quadrilateral is a parallelogram if its opposite sides are equal (page 128).

110. A quadrilateral is a parallelogram, if two sides are equal and parallel (page 129).

111. The diagonals of a parallelogram bisect each other (page 130).

114. If three or more parallel lines cut off equal lengths on one transversal, they cut off equal lengths on every transversal (page 138).

147. A radius perpendicular to a chord bisects the chord and its arc (page 162). (This may include the proof of corollary 148.)

149. In a circle or in equal circles, equal chords are equally distant from the center (page 164).

150. In a circle or in equal circles, chords equally distant from the center are equal (page 164).

157. Tangents to a circle from a point are equal (page 170).

165. An inscribed angle is measured by half its intercepted arc (page 182).

170. An angle formed by a tangent and a chord is measured by half its intercepted arc (page 188).

171. An angle formed by two chords intersecting inside a circle is measured by half the sum of its opposite intercepted arcs (page 189).

172. An angle formed by two lines intersecting outside a circle, and which meet the circle, is measured by half the difference of the intercepted arcs (page 190).

189. The perpendicular bisector of a line segment is the locus of a point equally distant from the ends of the segment (page 220). (This includes the proof of 59 and 60.)

190. The bisector of an angle is the locus of a point equally distant from the sides of the angle (page 220). (This includes the proof of 97 and 98.)

212. A line that divides two sides of a triangle proportionally is parallel to the third side (page 240).

216. Two triangles are similar, if two angles of one equal respectively two angles of the other (page 246). (*Note:* As stated in the syllabus, the proposition is worded " three angles " instead of " two angles," but of course the third angle is not needed in the hypothesis.)

220. Two triangles are similar, if an angle of one equals an angle of the other and the including sides are proportional (page 247).

222. Two triangles are similar, if the three sides of one are respectively proportional to the three sides of the other (page 249).

225. In a right triangle, the altitude on the hypotenuse is the mean proportional between the segments of the hypotenuse (page 266). (The remainder of this proposition is not required.)

227. In a *right triangle*, the square of the hypotenuse equals the sum of the squares of the legs (page 265).

229. If two chords intersect, the product of the segments of one equals the product of the segments of the other (page 277).

231. If a tangent and a secant are drawn from a point outside a circle, the tangent is the mean proportional between the secant and its external segment (page 279).

250. The area of a parallelogram equals the product of its base and altitude (page 316).

254. The area of a triangle equals half the product of its base and altitude (page 318).

260. The area of a trapezoid equals half its altitude times the sum of its bases (page 327).

261. The areas of similar triangles are to each other as the squares of corresponding sides (page 330).

278. A circle can be circumscribed about any regular polygon (page 350).

290. The area of a regular polygon equals half the product of its perimeter and its apothem (page 365).

294. If a circle is divided into any number of equal arcs, the chords of these arcs form a regular inscribed polygon (page 374).

300. If two sides of a triangle are unequal, the angle opposite the longer side is greater than the angle opposite the shorter side (page 389).

301. If two angles of a triangle are unequal, the side opposite the greater angle is longer than the side opposite the smaller angle (page 390).

Note especially that among the propositions required by the New York State syllabus are 111, 157, 170, 171, 172, 231, 278, 300 and 301, none of which is on the National Committee list and consequently not in black face type in this text.

In the case of problems of construction, the pupil is expected to be able to work out with ruler and compasses those listed below and other simple original constructions based upon these. He will not be required to give proofs of constructions on the examination unless such proofs are specifically called for in the questions.

63. The perpendicular bisector of a given line segment can be constructed (page 76).

64. Through a given point, a perpendicular to a given line can be constructed (page 77).

65. A given angle can be bisected (page 79).

66. An angle equal to a given angle can be constructed at a given point on a given line (page 80).

77. Through a given outside point a line parallel to a given line can be constructed (page 92).

116. A given line segment can be divided into any number of equal parts (page 139). (Note that this construction is required by the New York State syllabus, although not on the National Committee list and therefore in light face type in this text.)

174. A circle can be circumscribed about a given triangle (page 201).

177. A circle can be inscribed in a given triangle (page 203).

178. Through a given point, on or outside a circle, a tangent to the circle can be constructed (page 205).

180. Given its three sides, a triangle can be constructed (page 207).

181. Given two sides and the included angle, a triangle can be constructed (page 208).

182. Given two angles whose sum is less than a straight angle and the included side, a triangle can be constructed (page 209).

213. A fourth proportional to three given line segments can be constructed (page 242).

214. A line segment can be divided into parts proportional to two given segments (page 243).

229. A mean proportional between two given line segments can be constructed (page 277).

276. A triangle equal in area to a given polygon can be constructed (page 354). (For transforming a polygon into a square, use the methods of 263 after transforming it into a triangle.)

296. A square can be inscribed in a circle (page 353).

297. A regular hexagon can be inscribed in a circle (page 377).

In addition to this the pupil should be able to use the formulas given on pages 413 and 414.

APPENDIX B

SPECIMEN REGENTS EXAMINATIONS

THE UNIVERSITY OF THE STATE OF NEW YORK

June, 1936

GROUP I

Answer all questions in this group. Each correct answer will receive $2\frac{1}{2}$ credits. No partial credit will be allowed. Each answer must be reduced to its simplest form.

1 The area of a square is 16 square inches. Find the area of the inscribed circle. [Answer may be left in terms of π.] *Ans.....*

2 The area of an equilateral triangle is $25\sqrt{3}$. Find a side of the triangle. *Ans.....*

3 How many degrees are there in each interior angle of a regular octagon (eight-sided polygon)? *Ans.....*

4 Two angles of a triangle are 40° and 60°; how many degrees are there in the obtuse angle formed by the bisectors of these two angles? *Ans.....*

5 In triangle ABC, angle $C=90°$, $AB=25$, angle $A=55°$. Find AC correct to the nearest *tenth*. *Ans.....*

6 In a circle whose radius is 5″, a chord is drawn perpendicular to a diameter and 3″ from the center. What is the length of the chord in inches? *Ans.....*

7 Triangle ABC is inscribed in a circle. Angle $A=80°$, angle $B=70°$, angle $C=30°$. Which side of triangle ABC is nearest the center of the circle? *Ans.....*

8 Given triangle ABC with D a point on AB and E a point on BC such that DE is parallel to AC; if $BD=8$, $DA=6$ and $EC=9$, find BE. *Ans.....*

9 Two sides of a triangle are 10 and 14 and the angle included between these sides is 30°; what is the altitude on side 14? *Ans.....*

10 If the vertices of an inscribed triangle divide the circle into three arcs in the ratio $3:4:5$, how many degrees are there in the largest angle of the triangle? *Ans.....*

11 In parallelogram $ABCD$, angle B is twice angle A. How many degrees are there in angle A? *Ans.....*

12 The circumference of a circle is 8π. Find the radius of the circle. *Ans.....*

13 Two angles that are both equal and supplementary must be (*a*) adjacent, (*b*) acute or (*c*) right. Which is correct, (*a*), (*b*) or (*c*)? *Ans.....*

14 If a point is equidistant from the sides of a triangle, it must be the intersection of the three (*a*) altitudes, (*b*) medians or (*c*) angle bisectors. Which is correct, (*a*), (*b*) or (*c*)? *Ans.....*

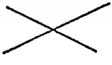
15 If each side of a triangle is multiplied by 2, then its area is multiplied by (*a*) 2, (*b*) 4 or (*c*) 6. Which is correct, (*a*), (*b*) or (*c*)? *Ans.....*

16 AB is one of the bases of trapezoid $ABCD$, and AC and BD are the diagonals; then triangles BCD and ACD are (*a*) similar, (*b*) congruent or (*c*) equal in area. Which is correct, (*a*), (*b*) or (*c*)? *Ans.....*

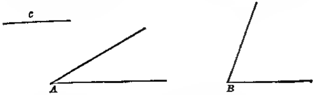
17 The vertices of the right angles of all right triangles having a common hypotenuse lie on (*a*) a line parallel to the hypotenuse, (*b*) a circle whose diameter is the hypotenuse, or (*c*) a semicircle whose diameter is the hypotenuse. Which is correct, (*a*), (*b*) or (*c*)? *Ans.....*

Directions (questions 18–20)—Leave all construction lines on the paper.

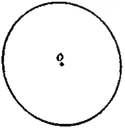
18 Construct the locus of points equidistant from the two intersecting lines at the right.



19 Construct triangle ABC, having given angle A, angle B and the included side c.



20 Inscribe an equilateral triangle in circle O.



Group II

Answer three questions from this group.

21 Prove that an angle formed by two chords intersecting within a circle is measured by one half the sum of the intercepted arcs. [10]

22 Prove that two right triangles are congruent if the hypotenuse and a leg of one are equal to the hypotenuse and a leg of the other. [10]

23 Angle ABC is inscribed in a circle. Chord BD bisects angle ABC and chord DE is drawn parallel to AB. Prove that chord DE equals chord BC. [10]

24 Two tangents from the external point A touch circle O at points B and C respectively. Lines OA, OC and BC are drawn.

BK is a line from B perpendicular to line CO extended to K. Prove: (a) OA is perpendicular to BC [5], (b) triangle ACO is similar to triangle BKC [5].

25 At the ends of a diameter of a circle, tangents to the circle are drawn. At any point on the circle a third tangent is drawn terminating in the other two tangents. Prove that the area of the trapezoid thus formed is equal to one half the product of the diameter and this third tangent. [10]

Group III

Answer two questions from this group.

26 In a right triangle whose shortest side is 30, the altitude upon the hypotenuse is 24. Find the segments of the hypotenuse made by the altitude. [10]

27 The perimeter of a regular pentagon is 50 inches. Find its area correct to the *nearest square inch.* [Use numerical trigonometry.] [10]

28 Construct one side of the square whose area will be equal to the area of a given triangle ABC. [Leave all construction lines on the paper.] [10]

January, 1937

Group I

Answer all questions in this group. Each correct answer will receive $2\frac{1}{2}$ credits. No partial credit will be allowed. Each answer must be reduced to its simplest form.

Directions (questions 1–9)—Write on the dotted line at the right of each question the expression which when inserted in the corresponding blank will make the statement true.

1 If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal must be . . .

Ans..

2 One angle of a right triangle is 60°. If the shortest side of the triangle is 8 inches, the hypotenuse is . . . inches. *Ans.....*

3 The sum of the interior angles of a polygon of five sides is . . . degrees. *Ans.....*

4 In a right triangle, the segments of the hypotenuse made by drawing the altitude upon the hypotenuse are 2 inches and 8 inches. The length of this altitude is . . . inches. *Ans.....*

5 A point is 6 inches from a circle whose radius is 9 inches. The length of the tangent from this point to the circle is . . . inches. *Ans.....*

6 The area of a square inscribed in a circle whose diameter is 10 inches is . . square inches. *Ans.....*

7 A cross section of an irrigation ditch has the form of a trapezoid. The upper base of the trapezoid is 30 feet, the lower base is 12 feet and the height is 8 feet. The area of the cross section is . square feet. *Ans.....*

8 Two circles have radii of 5 feet and 12 feet respectively. The radius of a circle whose area is equal to the sum of the areas of these two circles is . . . feet. *Ans.....*

9 Point P lies between two parallel lines a and b, which are 3 inches apart. The number of points that are equidistant from a and b and 2 inches from P is . . *Ans.....*

Directions (questions 10–13)—Indicate the correct answer to each of the following questions by writing the letter *a*, *b* or *c* in the space at the right.

10 A circle can always be circumscribed about a quadrilateral if the opposite angles of the quadrilateral are (*a*) complementary, (*b*) supplementary or (*c*) equal. *Ans.....*

11 Two circles are drawn so that they have four common tangents. The line segment joining the centers is (*a*) equal to the sum of the radii, (*b*) greater than the sum of the radii or (*c*) less than the sum of the radii. *Ans.....*

12 Corresponding sides in two similar polygons are in the ratio 1 : 4. The area of the larger polygon is (*a*) twice, (*b*) four times or (*c*) sixteen times, the area of the smaller polygon. *Ans*.....

13 A regular polygon and a triangle have equal areas. If the perimeter of the polygon is 10 and the base of the triangle is 8, the apothem of the polygon is (*a*) equal to, (*b*) greater than or (*c*) less than, the height of the triangle. *Ans*.....

Directions (questions 14–18)—Indicate whether each of the following statements is *always* true, *sometimes* true or *never* true by writing the word *always*, *sometimes* or *never* on the dotted line at the right.

14 If two rectangles have equal bases, their areas are to each other as their altitudes. *Ans*.....

15 If the three sides of a triangle are unequal, the altitude upon any side is equal to the median upon that side. *Ans*.....

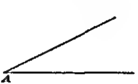
16 Similar triangles inscribed in the same circle or in equal circles have their corresponding sides equal. *Ans*.....

17 If a circle is circumscribed about a triangle, the center of the circle lies inside the triangle. *Ans*.....

18 If an altitude of an equilateral triangle is represented by $3x$, then x represents the radius of the inscribed circle. *Ans*.....

Directions (questions 19–20)—Leave all construction lines on the paper.

19 Construct the locus of the centers of circles each of which is tangent to both sides of the given angle A.



20 Construct the altitude of triangle ABC upon side AB.

Group II

Answer three questions from this group.

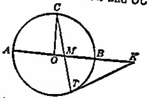
21 Prove that if the opposite sides of a quadrilateral are equal, the figure is a parallelogram. [10]

22 Prove that if two triangles have an angle of one equal to an angle of the other and the sides including these angles proportional, the triangles are similar. [10]

23 Given the isosceles triangle ABC with AB and AC the equal sides, D a point on AB, and AF the altitude upon BC. A line through D perpendicular to AB meets BC, extended if necessary, at E.

Prove: $\dfrac{FC}{BD}=\dfrac{AC}{BE}$ [10]

24 Given a circle whose center is O with AB a diameter and OC a radius perpendicular to AB; a line is drawn from C through M, any point on AB, cutting the circle at T. A tangent to the circle at T meets the diameter AB extended at K. Prove that KM and KT are equal. [10]



25 *a* Transform a given quadrilateral into a triangle. [6]
b Transform the triangle obtained in answer to *a* into an isosceles triangle. [4]
[Show all construction lines in both *a* and *b*.]

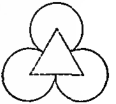
Group III

Answer two questions from this group.

26 The perimeter of the parallelogram $ABCD$ is 10 feet and the angle A is 45°. The altitude on base AB is represented by x.

a Express the side AD in terms of x. [$2\frac{1}{2}$]
b Express AB in terms of x. [$2\frac{1}{2}$]
c Express the area K of the parallelogram in terms of x. [$2\frac{1}{2}$]
d Find, correct to the *nearest square foot*, the value of K if the value of x is 2 feet. [$2\frac{1}{2}$]

27 A church window has the form of a trefoil as shown in the figure. The triangle used in the construction is equilateral, the vertices of the triangle are the centers of the circular arcs and the radius of each arc is one half the side of the triangle. The side of the triangle is 8 inches. Find (*a*) the perimeter of the figure [3], (*b*) the area of the figure [7]. [Answers may be left in terms of π and radicals.



28 AB is a diameter of a circle whose center is O. On OB extended a point P is taken 10 inches from O. Through P a secant is drawn intersecting the circle at C and D so that the arc $BC = 10°$ and the arc $AD = 60°$. Find, correct to the *nearest tenth of an inch*, the distance of the secant from the center of the circle. [Use numerical trigonometry.] [10]

June, 1937

Group I

Answer all questions in this group. Each correct answer will receive $2\frac{1}{2}$ credits. No partial credit will be allowed. Each answer must be reduced to its simplest form.

Directions (questions 1–8)—Write on the dotted line at the right of each question the expression which when inserted in the corresponding blank will make the statement true.

1 *An angle formed by two chords intersecting within* the circle is equal in degrees to one half the . . . of the intercepted arcs. *Ans.....*

2 The bisectors of the angles of a triangle meet in a point which is equidistant from the three . . . of the triangle. *Ans.....*

3 If the bases of a trapezoid are 12 inches and 18 inches, the length of the line segment joining the midpoints of the nonparallel sides is . . . inches. *Ans.....*

4 The area K of a trapezoid in terms of its altitude h and its bases b and b' is given by the formula $K =$ . . . . *Ans.....*

5 The area K of an equilateral triangle whose side is a is given by the formula $K =$ . . . . *Ans*. . . . .

6 If each exterior angle of a polygon is 40°, the number of sides of the polygon is . . . . *Ans*. . . . .

7 If an acute angle of a parallelogram is 60°, an obtuse angle of the parallelogram is . . . degrees. *Ans*. . . . .

8 A tangent and a secant are drawn to a circle from a point outside the circle. If the tangent is 14 inches long and the secant is 28 inches long, the length of the external segment of the secant is . . . inches. *Ans*. . . . .

Directions (questions 9–13)—Indicate whether each of the following statements is *always* true, *sometimes* true or *never* true by writing the word *always*, *sometimes* or *never* on the dotted line at the right.

9 The area of a rhombus is equal to one half the product of its diagonals. *Ans*. . . . .

10 If the angle between two tangents to a circle is 60°, the triangle formed by these tangents and the chord joining the points of contact is equiangular. *Ans*. . . . .

11 A line passing through the midpoint of a chord of a circle passes through the center of the circle. *Ans*. . . . .

12 If two angles of a triangle are 30° and 60°, the side opposite the 60° angle is twice the side opposite the 30° angle. *Ans*. . . . .

13 If C is the midpoint of the arc AB of a circle, then the chord AB is twice the chord AC. *Ans*. . . .

Directions (questions 14–17)—Indicate the correct answer to each of the following questions by writing the letter a, b, or c in the space at the right.

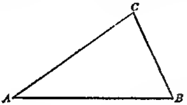
14 If two angles of one triangle are equal respectively to two angles of another triangle, the triangles must be (*a*) congruent, (*b*) similar or (*c*) equal. *Ans*. . . . .

15 The number of points which are equidistant from two given parallel lines and, at the same time, are equidistant from two given points on one of these lines is (*a*) one, (*b*) two or (*c*) three. *Ans*. . . . .

16 If the radius of a circle is increased by x, the circumference of the circle is increased by (*a*) x, (*b*) $2x$ or (*c*) $2\pi x$. *Ans.....*

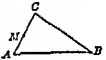
17 The areas of any two regular polygons which have equal perimeters are to each other as (*a*) their apothems, (*b*) the squares of their apothems or (*c*) their perimeters. *Ans.....*

Directions (questions 18–20)—Leave all construction lines on the paper.

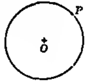


18 Find the center of tha circle that can be circumscribed about triangla ABC.

19 Through tha point M construct a line which will divide the sides AC and BC of triangla ABC proportionally.



20 Construct a tangent to the circle O at point P.

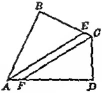


GROUP II

Answer three questions from this group.

21 Prove that if in the same circle or in equal circles two chords are equal, they are equidistant from the center. [10]

22 Prove that the areas of two similar triangles are to each other as the squares of any two corresponding sides. [10]



23 In the figure at the right, $ABCD$ is a quadrilateral, angles B and D are right angles, AE bisects angle A, and CF bisects angle C. Then AE is parallel to CF.

Below is a possible proof for this exercise.

Give a reason for *each* of the following statements:

1	$\angle DAB + \angle B + \angle BCD + \angle D = 360°$	[1]
2	$\angle B + \angle D = 180°$	[1]
3	$\angle DAB + \angle BCD = 180°$	[1]
4	$\angle BAE + \angle BCF = 90°$	[1]
5	$\angle BAE + \angle BEA = 90°$	[1]
6	$\angle BCF = \angle BEA$	[1]
7	$AE \parallel CF$	[1]

24 AB is a diameter of a circle and at B a tangent to the circle is drawn. From A, a line is drawn intersecting the circle at C and the tangent at D. Prove $AC : AB = AB : AD$ [10]

25 In the triangle ABC, D is any point in AC and E is any point in CB. DE is drawn. Prove that the sum of AC and CB is greater than the sum of AD, DE and EB. [10]

Group III

Answer two questions from this group.

26 The radius of a circular flower bed is 30 feet and this bed is surrounded by a path 3 feet wide. Find the cost of paving the path at 25 cents per square foot. (Use $\pi = \frac{22}{7}$) [10]

27 Given two lines m and n intersecting each other at right angles, and point P on one of these lines

a Describe completely the locus of points which are at a given distance s from P. [2]

b Describe completely the locus of points equidistant from m and n. [3]

c How many points are there which will satisfy both conditions given in *a* and *b* if P is 4 inches from the intersection of m and n and

(1) s is 5 inches long [2]

(2) s is 1 inch long [2]

28 The altitude of a triangle is 12 inches and it divides the vertex angle into two angles of 20° and 45°.

a Find the lengths of the segments of the base. [4]

b Find, correct to the *nearest square inch*, the area of the triangle. [4]

January, 1938

Group I

Answer all questions in this group. Each correct answer will receive $2\frac{1}{2}$ credits. No partial credit will be allowed. Each answer must be reduced to its simplest form.

Directions (questions 1–12)—Write on the dotted line at the right of each question the expression which when inserted in the corresponding blank will make the statement true.

1 The locus of points equidistant from the sides of an angle is the . . . of that angle. 1......

2 The bisectors of two complementary adjacent angles form an angle of . . . degrees. 2......

3 The sum of the interior angles of a polygon of *seven* sides is . . . straight angles. 3......

4 An angle formed by a tangent and a secant intersecting outside the circle is measured by one half the . . . of the intercepted arcs. 4......

5 If the areas of two similar triangles are in the ratio 1 : 25, corresponding sides of the triangles are in the ratio . . . . 5. . . . . .

6 The bases of a trapezoid are 8 inches and 10 inches and the area is 54 square inches. The altitude of the trapezoid is . . . inches. 6. . . . . .

7 The segments of one of two chords intersecting within a circle are r and s. If one segment of the other chord is m, the length of the other segment in terms of r, s and m is . . . . 7. . . . . .

8 In the right triangle ABC, hypotenuse AB is 20 inches and angle A is 54°; the length of AC, correct to the *nearest inch*, is . . . inches. 8. . . . . .

9 If one angle of a right triangle is 60° and the hypotenuse is 2, the length of the side opposite the 60° angle is . . . . [Answer may be left in radical form.] 9. . . . . .

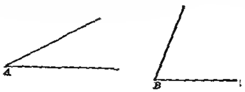
10 If the diagonals of a rhombus are 10 and 24, a side of the rhombus is . . . . 10. . . . . .

11 The centers of all circles tangent to the same line at the same point lie on a line which is (*a*) parallel to the given line or (*b*) perpendicular to the given line. The correct answer is . . . . [Answer *a* or *b*.] 11. . . . . .

12 The diagonal d of a square is equal to the side s multiplied by (*a*) $\sqrt{2}$, (*b*) $\sqrt{3}$ or (*c*) 2. The correct answer is . . . . [Answer *a*, *b* or *c*.] 12. . . . . .

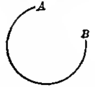
Directions (questions 13–15)—Leave all construction lines on the paper.

13 If the angles A and B are two angles of a triangle, find by construction the third angle of the triangle.



14 In the space below construct an angle of 30°.

15 Find by construction the center of the circle of which the arc AB is a part.



Directions (questions 16–20)—Indicate whether each of the following statements is *always* true, *sometimes* true or *never* true by writing the word *always*, *sometimes* or *never* on the dotted line at the right.

16 The diagonals of a rectangle are equal. 16......

17 The opposite angles of a quadrilateral are supplementary. 17......

18 If the radius of a circle is multiplied by k, the circumference is multiplied by k. 18......

19 An equilateral polygon is a regular polygon. 19......

20 The altitude upon any side of a triangle is greater than the median drawn to that side. 20......

Group II

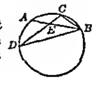
Answer three questions from this group.

21 Prove that if two sides of a quadrilateral are equal and parallel, the figure is a parallelogram. [10]

22 Prove that an angle formed by a tangent and a chord drawn from the point of contact is measured by one half the intercepted arc. [10]

23 Prove that the base angles of an isosceles trapezoid are equal. [10]

24 In the figure at the right, C is the midpoint of the arc AB. Chords AB and CD intersect in E and chords CB and BD are drawn. Prove that $CD \times CE = (CB)^2$ [10]



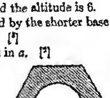
25 Given a right triangle whose legs are a and b; transform the triangle into a rectangle whose base is a given line segment m. [10]

Group III

Answer two questions from this group.

26 The bases of a trapezoid are 7 and 10 and the altitude is 6.

- *a* Find the altitude of the triangle formed by the shorter base and the nonparallel sides produced. [5]
- *b* Find the area of the triangle described in *a*. [5]



27 The figure at the right represents the cross section of a hexagonal nut. Assuming that the diameter of the circle and the side of the regular hexagon are each 2 inches, find, correct to the *nearest square inch*, the area of the cross section (the shaded portion). [10]

28 Given a parallelogram with two adjacent sides a and b and included angle C

- *a* What change takes place in the area of the parallelogram if a and b remain constant and angle C increases (1) from $0°$ to $90°$, (2) from $90°$ to $180°$? [1, 1]
- *b* Find, correct to the *nearest tenth*, the area of the parallelogram if $a = 4$, $b = 5$ and angle $C = 52°$ [8]

June, 1938

Group I

Answer all questions in this group. Each correct answer will receive $2\frac{1}{2}$ credits. No partial credit will be allowed. Each answer must be reduced to its simplest form.

Directions (questions 1–8)—Write on the dotted line at the right of each question the expression which when inserted in the corresponding blank will make the statement true.

1 The angle formed by two secants intersecting outside a circle is measured by one half the . . . of the intercepted arcs. 1. . . . . .

2 The area of a circle is 25π square inches. The radius of this circle is . . . inches. 2......

3 If the vertex angle of an isosceles triangle is $40°$, the number of degrees in either of the exterior angles formed by extending the base is . . . . 3......

4 If each exterior angle of a regular polygon contains $45°$, the number of sides of the polygon is . . . . 4......

5 In triangle ABC angle $B=30°$ and side $AB=$ 6 inches. The length of the altitude AD upon side BC is . . . inches. 5......

6 A pair of corresponding altitudes of two similar triangles are 4 inches and 2 inches. The area of the larger triangle is . . . times the area of the smaller. 6......

7 The radius of a circle is 13 inches and a chord of this circle is 10 inches. The distance of this chord from the center of the circle is . . . inches. 7......

8 At a point 100 feet from the foot of a flagpole the angle of elevation of the top of the pole is $31°$. The height of the flagpole, correct to the *nearest foot*, is . . . feet. 8......

Directions (questions 9–12)—Indicate the correct answer to each of the following questions by writing the letter *a* or *b* on the line at the right.

9 The area of a rhombus is equal to (*a*) the product of two diagonals or (*b*) one half the product of the two diagonals. 9......

10 An axiom is a statement which (*a*) is to be proved or (*b*) is accepted without proof. 10......

11 The circumference of a circle whose radius is 4, is (*a*) 8π or (*b*) 16π. 11......

12 The locus of the centers of all circles tangent to each of two parallel lines is (*a*) one straight line or (*b*) two straight lines. 12......

Directions (questions 13–17)—Indicate whether each of the following statements is *always* true, *sometimes* true or *never* true by

25 Given a right triangle whose legs are a and b; transform the triangle into a rectangle whose base is a given line segment m. [10]

Group III

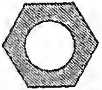
Answer two questions from this group.

26 The bases of a trapezoid are 7 and 10 and the altitude is 6.

a Find the altitude of the triangle formed by the shorter base and the nonparallel sides produced. [3]

b Find the area of the triangle described in *a*. [7]

27 The figure at the right represents the cross section of a hexagonal nut. Assuming that the diameter of the circle and the side of the regular hexagon are each 2 inches, find, correct to the *nearest square inch*, the area of the cross section (the shaded portion). [10]



28 Given a parallelogram with two adjacent sides a and b and included angle C

a What change takes place in the area of the parallelogram if a and b remain constant and angle C increases (1) from $0°$ to $90°$, (2) from $90°$ to $180°$? [1, 1]

b Find, correct to the *nearest tenth*, the area of the parallelogram if $a=4$, $b=5$ and angle $C=52°$ [8]

June, 1938

Group I

Answer all questions in this group. Each correct answer will receive $2\frac{1}{2}$ credits. No partial credit will be allowed. Each answer must be reduced to its simplest form.

Directions (questions 1–8)—Write on the dotted line at the right of each question the expression which when inserted in the corresponding blank will make the statement true.

1 The angle formed by two secants intersecting outside a circle is measured by one half the . . . of the intercepted arcs. 1......

2 The area of a circle is 25π square inches. The radius of this circle is . . . inches. 2......

3 If the vertex angle of an isosceles triangle is 40°, the number of degrees in either of the exterior angles formed by extending the base is . . . . 3......

4 If each exterior angle of a regular polygon contains 45°, the number of sides of the polygon is . . . . 4......

5 In triangle ABC angle $B=30°$ and side $AB=$ 6 inches. The length of the altitude AD upon side BC is . . . inches. 5......

6 A pair of corresponding altitudes of two similar triangles are 4 inches and 2 inches. The area of the larger triangle is . . . times the area of the smaller. 6......

7 The radius of a circle is 13 inches and a chord of this circle is 10 inches. The distance of this chord from the center of the circle is . . . inches. 7......

8 At a point 100 feet from the foot of a flagpole the angle of elevation of the top of the pole is 31°. The height of the flagpole, correct to the *nearest foot*, is . . . feet. 8......

Directions (questions 9–12)—Indicate the correct answer to each of the following questions by writing the letter *a* or *b* on the line at the right.

9 The area of a rhombus is equal to (*a*) the product of two diagonals or (*b*) one half the product of the two diagonals. 9......

10 An axiom is a statement which (*a*) is to be proved or (*b*) is accepted without proof. 10......

11 The circumference of a circle whose radius is 4, is (*a*) 8π or (*b*) 16π. 11......

12 The locus of the centers of all circles tangent to each of two parallel lines is (*a*) one straight line or (*b*) two straight lines. 12......

Directions (questions 13–17)—Indicate whether each of the following statements is *always* true, *sometimes* true or *never* true by

writing the word *always*, *sometimes*, or *never* on the dotted line at the right.

13 When two straight lines are cut by a transversal, if the two interior angles on the same side of the transversal are supplementary, the two straight lines are parallel. 13......

14 If two angles and a side of one triangle are equal to the corresponding parts of another triangle, then the triangles are congruent. 14......

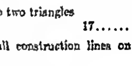
15 A triangle can be constructed having sides 8 inches, 12 inches and 3 inches. 15......

16 If a theorem is true, a converse of the theorem is true. 16......

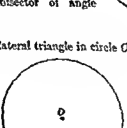
17 A median of a triangle divides it into two triangles which are equal in area. 17......

Directions (questions 18-20)—Leave all construction lines on the paper.

18 Construct the bisector of angle *ABC*.



19 Inscribe an equilateral triangle in circle *O*.



20 Find by construction the two points which are equidistant from points *A* and *B* and also at the given distance *s* from point *C*.

Group II

Answer three questions from this group.

21 Prove that the diagonals of a parallelogram bisect each other. [10]

22 Prove that if from a point outside a circle a tangent and a secant are drawn to the circle, the tangent is the mean proportional between the secant and its external segment. [10]

23 Given triangle ABC with $AB=AC$; D is any point between B and C in BC and line segment AD is drawn. Prove that $AB>AD$. [10]

24 PA and PB are tangents drawn to a circle whose center is O, A and B being the points of tangency. Line segment PO is drawn and extended to meet the circle at C. Prove that arc AC = arc BC. [The use of original exercises as reasons will not be allowed in this proof.] [10]

25 In the figure at the right, AB is the diameter of a semicircle. A point P moves along the arc from A to B and chords PA and PB are drawn.

a Using the expression *increases, decreases* or *remains the same,* indicate the change, if any, that takes place in (1) angle PAB, (2) angle APB and (3) angle PBA. [6]

b At what position of P will the altitude of triangle APB on base AB be greatest? [2]

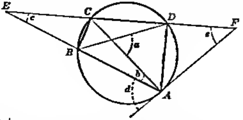
c At what position of P will the area of triangle APB be greatest? [2]

Group III

Answer two questions from this group.

26 In the drawing below, $ABCD$ is a quadrilateral inscribed in a circle; arc $AB=135°$, arc $BC=40°$ and arc $CD=100°$. Chords AC and BD are drawn; also chords AB and DC are extended to meet at E and the tangent at A meets CD extended at F. Find

the number of degrees in angle a, angle b, angle c, angle d and angle e. [10]



27 Given two equal circles with a square inscribed in one and a regular hexagon in the other; if the radius of each circle is 14 inches, find, correct to the *nearest tenth of a square inch*, the difference in the areas of the regular hexagon and the square. [Use $\sqrt{3}=1.73$] [10]

28 The bases of an isosceles trapezoid are 8 and 28. One base angle is 53°.

a Find, correct to the *nearest tenth*, the altitude of the trapezoid. [7]

b Find, correct to the *nearest integer*, the area of the trapezoid. [3]

January, 1939

Group I

Answer all questions in this group. Each correct answer will receive 2 credits. No partial credit will be allowed. Each answer must be reduced to its simplest form.

Directions (questions 1–17)—Indicate the correct answer to each of the following questions by writing on the dotted line at the right the letter a, b or c.

1 The diagonals of a rectangle are always (a) equal to each other, (b) perpendicular to each other or (c) bisectors of the angles through which they pass. 1......

2 If two adjacent angles have their exterior sides in the same straight line, they are always (a) equal, (b) complementary or (c) supplementary. 2......

3 The sum of the exterior angles of a polygon of n sides is (*a*) n straight angles, (*b*) 2 straight angles or (*c*) $(n-2)$ straight angles. 3......

4 If two parallel lines are cut by a transversal, the corresponding angles are always (*a*) supplementary, (*b*) equal or (*c*) acute. 4......

5 In triangle ABC, AD is an altitude and AM is a median. If AB and AC are unequal, (*a*) $AM=AD$, (*b*) $AM>AD$ or (*c*) $AM<AD$. 5......

6 A median of a triangle divides it into two triangles which are always (*a*) congruent, (*b*) similar or (*c*) equal in area. 6......

7 The area of a rhombus is equal to (*a*) one half the sum of its diagonals, (*b*) one half the product of its diagonals or (*c*) the product of its diagonals. 7......

8 If the corresponding sides of two similar triangles are in the ratio 1 : 2, the areas of the two triangles are in the ratio (*a*) 1 : 4, (*b*) 1 : 2 or (*c*) $1:\sqrt{2}$ 8......

9 If the segments of one of two chords, not diameters, intersecting within a circle, are r and s and the segments of the other chord are v and w, then (*a*) $r\times s=v\times w$, (*b*) $r+s=v+w$ or (*c*) $\frac{r}{s}=\frac{v}{w}$ 9......

10 If from a point outside a circle a tangent and a secant are drawn to the circle, the tangent is the mean proportional between (*a*) the whole secant and its internal segment, (*b*) the whole secant and its external segment or (*c*) the external and the internal segments of the secant. 10......

11 If in the right triangle ABC, AB is the hypotenuse and CD is the altitude upon the hypotenuse, then (*a*) $(CD)^2=AD\times DB$, (*b*) $(CD)^2=AB\times AD$ or (*c*) $(CD)^2=AC\times CB$ 11......

12 Similar polygons are defined as polygons which have (*a*) their corresponding angles equal, (*b*) their corresponding sides proportional or (*c*) their corresponding angles equal and their corresponding sides proportional. 12......

13 If in triangle ABC, BD is the altitude upon AC, then BD equals (a) $AB \times \sin A$, (b) $AB \times \tan A$ or (c) $AB \times \cos A$. 13......

14 The center of the circle circumscribed about a triangle is always the intersection of (a) the bisectors of two angles, (b) two altitudes or (c) the perpendicular bisectors of two sides. 14......

15 An inscribed angle of 80° intercepts an arc of (a) 80°, (b) 40° or (c) 160°. 15......

16 The locus of the centers of all circles which pass through two given points is (a) a circle, (b) a straight line or (c) a point. 16......

17 Converses of propositions are (a) always true, (b) sometimes true or (c) never true. 17......

Directions (questions 18–22)—Write on the dotted line at the right of each question the expression which when inserted in the corresponding blank will make the statement true.

18 The formula for the altitude h of an equilateral triangle in terms of its side s is $h =$ . . . . 18......

19 The formula for the diagonal d of a square in terms of its side s is $d =$ . . . . 19......

20 The formula for the circumference c of a circle in terms of its radius r is $c =$ . . . . 20......

21 The hypotenuse of a right triangle is 17 and one leg is 15; the length of the other leg is . . . . 21......

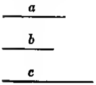
22 If the base of a triangle is 24, the length of the line segment joining the midpoints of the other two sides is . . . . 22......

Directions (questions 23–25)—Leave all construction lines on your paper.

23 Find by construction the locus of points within angle ABC and equidistant from the sides of the angle.

24 Construct the altitude of triangle ABC upon side AB.

25 Construct the fourth proportional to the three line segments a, b and c.



Group II

Answer two questions from this group.

26 Prove that an angle formed by two chords intersecting within a circle is measured by one half the sum of the intercepted arcs. [10]

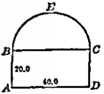
27 Prove that two right triangles are congruent if the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other. [10]

28 Triangle ABC is inscribed in a circle. The bisector of angle C intersects side AB at D and arc AB at E. Prove: $AC \times BC = CD \times CE$ [10]

Group III

Answer two questions from this group.

29 The accompanying figure $ABECD$ represents the cross section of an underground tunnel. $ABCD$ is a rectangle 40.0 feet by 20.0 feet, surmounted by the semicircle BEC. Find, correct to the *nearest square foot*, the area of the cross section. (Use $\pi = 3.14$) [10]



30 The area of an equilateral triangle is equal to that of a trapezoid whose bases are 4 and 14 and whose altitude is $4\sqrt{3}$. Find a side of the triangle. [10]

31 A side of a regular pentagon is 8. Find its area correct to the *nearest integer*. [10]

Group IV

Answer one question from this group.

32 Consider *each* of the following statements and tell whether it is *always* true, *sometimes* true or *never* true. *Give reasons for your answers.*

a The area of a rectangle is equal to one half the product of its diagonals. [1, 1½]

b If two triangles have a side and any two angles of one equal to the corresponding parts of the other, the triangles are congruent. [1, 1½]

c If the radius of a circle is increased by x, then the circumference of the circle is increased by $2\pi x$. [1, 1½]

d If the legs of a right triangle are represented by a and b and the hypotenuse by c, then $c^2=(a+b)^2$ [1, 1½]

33 A square whose side is s and a rectangle whose base is $s+a$ have equal perimeters.

a Express the altitude of the rectangle in terms of s and a. [4]

b Express the area of the rectangle in terms of s and a. [2]

c Indicate whether the following statement is true or false: Of all rectangles which have equal perimeters, that which is equilateral has the greatest area. Give reasons for your answer. [1, 3]

June, 1939

Group I

Answer all questions in this group. Each correct answer will receive 2 credits. No partial credit will be allowed. Each answer must be reduced to its simplest form.

Directions (questions 1–11)—Indicate the correct answer to each question by writing on the dotted line at the right the letter *a*, *b* or *c*.

1 A central angle of 60° intercepts an arc of (*a*) 30°, (*b*) 60° or (*c*) 120°. 1......

2 The opposite angles of a quadrilateral inscribed in a circle are always (*a*) equal, (*b*) complementary or (*c*) supplementary. 2......

3 The difference between the supplement and the complement of an angle is (*a*) an acute angle, (*b*) a right angle or (*c*) an obtuse angle. 3......

4 If the areas of two similar triangles are in the ratio 1 : 4, then any two corresponding sides of these triangles are in the ratio (*a*) 1 : 4, (*b*) 1 : 2 or (*c*) 1 : 16. 4......

5 The center of the circle inscribed in a triangle is always the intersection of (*a*) the bisectors of two of its angles, (*b*) two of its altitudes or (*c*) the perpendicular bisectors of two of its sides. 5......

6 A circle can always be circumscribed about (*a*) an equiangular polygon, (*b*) an equilateral polygon or (*c*) a regular polygon. 6......

7 The altitude drawn to the hypotenuse of a right triangle divides the triangle into two triangles which are always (*a*) congruent, (*b*) similar or (*c*) equal in area. 7......

8 The sum of the interior angles of a polygon of n sides is (*a*) n straight angles, (*b*) 2 straight angles or (*c*) $(n-2)$ straight angles. 8......

9 Each of the following sets of numbers can be used as the sides of a triangle: (*a*) 4″, 8″, 9″; (*b*) 7″, 24″, 25″; (*c*) 6″, 9″, 10″. Which set would form a right triangle? 9......

10 If in triangle ABC, CD is the altitude upon AB, then (*a*) $CD = AD \sin A$, (*b*) $CD = AD \cos A$ or (*c*) $CD = AD \tan A$. 10......

11 The formula for the area A of an equilateral triangle in terms of its side s is (*a*) $A = \frac{s}{2}\sqrt{3}$, (*b*) $A = \frac{s^2}{2}\sqrt{3}$ or (*c*) $A = \frac{s^2}{4}\sqrt{3}$ 11......

Directions (questions 12–22)—Write on the dotted line at the right of each question the expression which when inserted in the corresponding blank will make the statement true.

12 AB, the hypotenuse of right triangle ABC, is 10. If angle A is 30°, then BC is . . . . 12......

13 ABC is an isosceles triangle in which AC equals BC and one of the angles is obtuse. The longest side is opposite angle . . . . 13......

14 Two chords intersect within a circle. The segments of one chord are 8 and 3. If one segment of the second chord is 6, then the other segment is . . . . 14......

15 If the diameter of a circle is 10, then the circumference in terms of π is . . . . 15......

16 The formula for the area A of a trapezoid in terms of its altitude h and its bases b and b' is $A =$ . . . 16......

17 The perimeter of a regular polygon is 24. If its apothem is 3, then its area is . . . 17......

18 In a right triangle the length of one leg is 20 inches and the length of the hypotenuse is 25 inches. The length of the longer segment of the hypotenuse made by the altitude upon it is . . . inches. 18......

19 The locus of the midpoints of all radii of a given circle is a . . . . 19......

20 If the diagonals of a parallelogram are unequal and bisect the angles through which they are drawn, then the figure must be a . . . . 20......

21 The radius of a circle is 6. The angle of a sector of this circle is 60°. The area of the sector in terms of π is . . . . 21......

22 AB is a diameter of a circle, AC is a chord and arc AC contains 100°. Angle BAC contains . . . degrees. 22......

Directions (questions 23–25)—Leave all construction lines on the paper.

23 Construct the locus of points equidistant from the two given points A and B.

A

$\cdot B$

24 Construct a line parallel to line AB through point C.

$C\cdot$

A——————B

25 Construct the mean proportional between line segments a and b.

a

b

Group II

Answer two questions from this group.

26 Prove that the diagonals of a parallelogram bisect each other. [10]

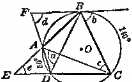
27 O is the midpoint of base AB of isosceles triangle ABC. AC and BC are extended through C to points E and D so that CE is equal to CD. Lines DO and EO are drawn. Prove: $DO = EO$ [10]

28 Prove that if two triangles have an angle of one equal to an angle of the other and the sides including these angles proportional, the triangles are similar. [10]

Group III

Answer two questions from this group.

29 $ABCD$ is a quadrilateral inscribed in circle O. Chord BA = chord CD, and BA and CD extended meet in point E. A tangent at B intersects DA extended in point F. Diagonals BD and AC are drawn. Arc $AD = 50°$ and arc $BC = 140°$. Find the number of degrees in angle a, angle b, angle c, angle d and angle e. [10]



30 In a circle a chord AB intercepts an arc of 100°. If the radius of the circle is 10.0 inches, find, correct to the *nearest inch*, the length of this chord. [10]

31 In the isosceles trapezoid $ABCD$, angle A is 45°; the longer base AB is 17 and the shorter base is 7. Find

a The area of the trapezoid [5]

b The length of the diagonal BD [5]

Group IV

Answer one question from this group.

32 Consider *each* of the following statements and tell whether it is *always* true, *sometimes* true or *never* true. *Give reasons for your answers.*

a A trapezoid inscribed in a circle is isosceles. [½, 2]

b The center of the circle circumscribed about a triangle lies within the triangle. [½, 2]

c If two triangles have two sides and an angle of one equal to the corresponding parts of the other, then the triangles are congruent. [½, 2]

d In triangle ABC, AB is greater than AC. If the bisectors of angles B and C intersect in D, then DC is greater than BD. [½, 2]

33 *a* Construct triangle ABC, given side b, the altitude on side b and the median to side b. [8]

b Is it possible to construct the triangle if the altitude is (1) greater than the median, (2) equal to the median? [1, 1]

INDEX